ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2=3$, then prove that :

$$3(a+b+c)+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 15$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\rightarrow(1)\Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=r^2s\rightarrow(2)$ and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\rightarrow(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab$$

$$\overset{\text{via (1) and (3)}}{=} s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\rightarrow(4)$$

$$\text{Now, }3(a+b+c)+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 15\overset{\because a^2+b^2+c^2=3}{\Leftrightarrow}$$

$$3\sum_{cyc}a+\frac{2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)}{3abc}\ge 5.\sqrt{3\sum_{cyc}a^2}$$

$$\overset{\text{via (1),(2),(3) and (4)}}{\Leftrightarrow}\left(3s+\frac{2(4Rr+r^2)(s^2-8Rr-2r^2)}{3r^2s}\right)^2\ge 75(s^2-8Rr-2r^2)$$

$$\Leftrightarrow(32R^2+88Rr-277r^2)s^4-rs^2(512R^3+960R^2r-2316Rr^2-631r^3)$$

$$+8r^2(4R+r)^4\overset{(*)}{\ge}0$$

$$\because 32R^2+88Rr-277r^2=(R-2r)(32R+152r)+27r^2\overset{\text{Euler}}{\ge}27r^2>0$$

$$\therefore\text{LHS of }(*)\overset{\text{Gerretsen}}{\ge}(32R^2+88Rr-277r^2)(16Rr-5r^2)s^2$$

$$-rs^2(512R^3+960R^2r-2316Rr^2-631r^3)+8r^2(4R+r)^4\overset{?}{\ge}0$$

$$\Leftrightarrow 9(8R-7r)(R-8r)s^2+2(4R+r)^4\underset{(**)}{\overset{?}{\ge}}0$$

$\boxed{\text{Case 1}}$ $R-8r\ge 0$ and then : LHS of $(**)\ge 2(4R+r)^4>0\Rightarrow(**)$ is true (strict inequality)

$\boxed{\text{Case 2}}$ $R-8r<0$ and then : LHS of $(**)=-9(8R-7r)(8r-R)s^2+2(4R+r)^4\overset{\text{Gerretsen}}{\ge}-9(8R-7r)(8r-R)(4R^2+4Rr+3r^2)+2(4R+r)^4\overset{?}{\ge}0$

$$\Leftrightarrow 800t^4-1756t^3-132t^2+131t+1514\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\Big((t-2)(800t^2+1444t+2444)+4131\Big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (**) \text{ is true} \therefore \text{combining both cases}, (**) \Rightarrow (*) \text{ is true } \forall\, \Delta\, ABC$$

$$\therefore 3(a+b+c)+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 15\ \forall\ a,b,c>0 \,\Big|\, a^2+b^2+c^2=3,$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $3a+\frac{2}{a}=\frac{a^2+9}{2}+\frac{(4-a)(a-1)^2}{2a} \geq \frac{a^2+9}{2}$, for all $a<2$.

Then:

$$3(a+b+c)+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\left(3a+\frac{2}{a}\right)+\left(3b+\frac{2}{b}\right)+\left(3c+\frac{2}{c}\right)$$

$$\geq \frac{a^2+b^2+c^2+3.9}{2}=15,$$

as desired. Equality holds iff $a=b=c=1$.