ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that :

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} + \sqrt{c^2 - ca + a^2}$$

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$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{a^2}{b} - a + b + \frac{b^2}{c} - b + c + \frac{c^2}{a} - c + a$$

$$= \frac{a^2 - ab + b^2}{b} + \frac{b^2 - bc + c^2}{b} + \frac{c^2 - ca + a^2}{a} \to (1)$$

$$\text{Now, } \sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} + \sqrt{c^2 - ca + a^2}$$

$$= \sum_{cyc} \sqrt{\frac{a^2 - ab + b^2}{b} . b} \overset{A-G}{\leq} \frac{1}{2} \sum_{cyc} \left(\frac{a^2 - ab + b^2}{b} + b \right)$$

$$\overset{A-G}{\leq} \sum_{cyc} \frac{a^2 - ab + b^2}{b} - \frac{1}{2} \sum_{cyc} \frac{2ab - ab}{b} + \frac{1}{2} \sum_{cyc} a$$

$$= \sum_{cyc} \frac{a^2 - ab + b^2}{b} - \frac{1}{2} \sum_{cyc} a + \frac{1}{2} \sum_{cyc} a \overset{via\,(1)}{=} \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

$$\therefore \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} + \sqrt{c^2 - ca + a^2}$$

$$\forall\, a, b, c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$