ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a + b + c = 3$, then prove that :

$$(a^3 + a)^b(b^3 + b)^c(c^3 + c)^a \le 8$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \to (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s - x)(s - y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4) \text{ and}$$

$$\sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\sum_{cyc} a \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R + r)^2 - 2s^2\right) \to (5)$$

$$\text{Now, } \sqrt[a+b+c]{(a^3 + a)^b(b^3 + b)^c(c^3 + c)^a}$$

$$= \sqrt[a+b+c]{(a^b b^c c^a)(a^2 + 1)^b(b^2 + 1)^c(c^2 + 1)^a}$$

$$\overset{\text{Weighted GM} \le \text{Weighted AM}}{\le} \frac{ab + bc + ca}{a + b + c} \cdot \frac{a^2b + b + b^2c + c + c^2a + a}{a + b + c}$$

$$\overset{\text{CBS}}{\le} \frac{\sum_{cyc} ab}{\sum_{cyc} a} \cdot \frac{\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)} + \sum_{cyc} a}{\sum_{cyc} a} \overset{?}{\le} \sqrt[a+b+c]{8} \overset{a+b+c=3}{=} 2$$

$$\Leftrightarrow \left(\sum_{cyc} ab\right)\left(\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)} + \sum_{cyc} a\right) \overset{?}{\le} 2\left(\sum_{cyc} a\right)^2$$

$$\overset{\because a+b+c=3}{\Leftrightarrow} \left(\sum_{cyc} ab\right)\left(\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)} + \left(\sum_{cyc} a\right) \cdot \frac{1}{9}\left(\sum_{cyc} a\right)^2\right) \overset{?}{\le} 2\left(\sum_{cyc} a\right)^2 \cdot \frac{1}{27}\left(\sum_{cyc} a\right)^3$$

$$\Leftrightarrow 2\left(\sum_{cyc} a\right)^5 - 3\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^3 \overset{?}{\geq} 27\left(\sum_{cyc} ab\right).\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)}$$

$$\Leftrightarrow \boxed{\left(2\left(\sum_{cyc} a\right)^5 - 3\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^3\right)^2 \overset{?}{\geq} 729\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} ab\right)^2}$$

$$\overset{\text{via } (1),(3),(4) \text{ and } (5)}{\Leftrightarrow} \left(2s^5 - 3(4Rr + r^2)s^3\right)^2$$

$$\overset{?}{\geq} 729(s^2 - 8Rr - 2r^2)\left(r^2((4R + r)^2 - 2s^2)\right)(4Rr + r^2)^2$$

$$\Leftrightarrow 4s^{10} - (48Rr + 12r^2)s^8 + r^2(144R^2 + 72Rr + 9r^2)s^6$$
$$+r^4(23328R^2 + 11664Rr + 1458r^2)s^4$$
$$-r^4(186624R^4 + 373248R^3r + 209952R^2r^2 + 46656Rr^3 + 3645r^4)s^2$$
$$+r^5\begin{pmatrix}1492992R^5 + 1866240R^4r + 933120R^3r^2 + 233280R^2r^3\\ +29160Rr^4 + 1458r^5\end{pmatrix}\boxed{\overset{?}{\geq}_{(*)}} 0 \text{ and}$$

$$\because 4(s^2 - 16Rr + 5r^2)^5 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \geq 4(s^2 - 16Rr + 5r^2)^5$$

$$\Leftrightarrow (272Rr - 112r^2)s^8 - r^2(10096R^2 - 6472Rr + 991r^2)s^6$$
$$+r^3(163840R^3 - 130272R^2r + 59664Rr^2 - 3542r^3)s^4$$
$$-r^4(1497344R^4 - 1265152R^3r + 977952R^2r^2 - 113344Rr^3 + 16145r^4)s^2$$
$$+r^5\begin{pmatrix}5687296R^5 - 4687360R^4r + 5029120R^3r^2 - 1046720R^2r^3\\ +229160Rr^4 - 11042r^5\end{pmatrix}\boxed{\overset{(**)}{\geq}} 0$$

$$\text{and} \because (272Rr - 112r^2)(s^2 - 16Rr + 5r^2)^4 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \geq (272Rr - 112r^2)(s^2 - 16Rr + 5r^2)^4$$

$$\Leftrightarrow (7312R^2 - 6136Rr + 1249r^2)s^6$$
$$-r(253952R^3 - 302880R^2r + 88656Rr^2 - 13258r^3)s^4$$
$$+r^2(2959104R^4 - 4747776R^3r + 2047968R^2r^2 - 560256Rr^3 + 39855r^4)s^2$$

$$\text{and} \because (7312R^2 - 6136Rr + 1249r^2)(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0,$$

$$-r^3\begin{pmatrix}12138496R^5 - 24934912R^4r + 14590720R^3r^2\\ -5430080R^2r^3 + 836840Rr^4 - 58958r^5\end{pmatrix}\boxed{\overset{(***)}{\geq}} 0$$

$$\therefore \text{in order to prove } (***), \text{it suffices to prove : LHS of } (***)$$
$$\geq (7312R^2 - 6136Rr + 1249r^2)(s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow (97024R^3 - 101328R^2r + 63336Rr^2 - 5477r^3)s^4$$
$$-r(2656512R^4 - 3474432R^3r + 2404944R^2r^2 - 499464Rr^3 + 53820r^4)s^2$$
$$+r^3\begin{pmatrix}17811456R^5 - 28276224R^4r + 22861824R^3r^2\\ -7643280R^2r^3 + 1428960Rr^4 - 97167r^5\end{pmatrix}\boxed{\overset{(****)}{\geq}} 0 \text{ and}$$

$$\because (97024R^3 - 101328R^2r + 63336Rr^2 - 5477r^3)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } (****), \text{it suffices to prove : LHS of } (****) \geq$$
$$(97024R^3 - 101328R^2r + 63336Rr^2 - 5477r^3)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow \boxed{(224128R^4 - 369152R^3r + 317544R^2r^2 - 154580Rr^3 + 475r^4)s^2 \overset{(*****)}{\geq} r(3513344R^5 - 6593792R^4r + 5995136R^3r^2 - 3212896R^2r^3 + 515380Rr^4 - 19879r^5)}$$

$$\text{Now, } (224128R^4 - 369152R^3r + 317544R^2r^2 - 154580Rr^3 + 475r^4)s^2 \overset{\text{Gerretsen}}{\geq}$$
$$(224128R^4 - 369152R^3r + 317544R^2r^2 - 154580Rr^3 + 475r^4)(16Rr - 5r^2)$$
$$\overset{?}{\geq} r\begin{pmatrix}3513344R^5 - 6593792R^4r + 5995136R^3r^2 - \\ 3212896R^2r^3 + 515380Rr^4 - 19879r^5\end{pmatrix}$$
$$\Leftrightarrow \boxed{9088t^5 - 54160t^4 + 116416t^3 - 106013t^2 + 33140t + 2188 \overset{?}{\geq} 0} \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)^2(184t^3 + 8904t^2(t-2) + 8832t + 547) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\Rightarrow (*****) \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore (a^3+a)^b(b^3+b)^c(c^3+c)^a \leq 8 \ \forall a,b,c > 0 \mid a+b+c = 3,$$
$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$a^3 + a = \frac{2a(a^2+1)}{2} \leq \frac{[2a + (a^2+1)]^2}{4.2} = \frac{(a+1)^4}{8} \quad \text{(and analogs)}.$$

Then

$$(a^3+a)^b(b^3+b)^c(c^3+c)^a \leq \frac{(a+1)^{4b}(b+1)^{4c}(c+1)^{4a}}{8^{b+c+a}}$$

$$\overset{Weighted\ AM-GM}{\leq} \frac{1}{8^3}\left(\frac{4b(a+1) + 4c(b+1) + 4a(c+1)}{4b+4c+4a}\right)^{4b+4c+4a} = \frac{1}{8^3}\left(\frac{ab+bc+ca}{3} + 1\right)^{12}$$
$$\leq \frac{1}{8^3}\left(\frac{(a+b+c)^2}{9} + 1\right)^{12} = \frac{1}{8^3}.2^{12} = 8.$$

Equality holds iff $a = b = c = 1$.