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If $a, \mathrm{b} > 0$ and $a + \mathrm{b} \geq \frac{1}{a^2} + \frac{1}{\mathrm{b}^2}$, then prove that :

$$\frac{a^2}{a+1} + \frac{\mathrm{b}^2}{\mathrm{b}+1} \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$a + \mathrm{b} \geq \frac{1}{a^2} + \frac{1}{\mathrm{b}^2} \overset{\text{Radon}}{\geq} \frac{(1+1)^3}{(a+\mathrm{b})^2} \Rightarrow a + \mathrm{b} \geq 2 \Rightarrow \mathrm{t} = x + \mathrm{y} \geq 4 \to (1)$$

$(x = a + 1, \mathrm{y} = \mathrm{b} + 1)$

$$\text{Now, } \frac{a^2}{a+1} + \frac{\mathrm{b}^2}{\mathrm{b}+1} \overset{?}{\geq} 1 \Leftrightarrow \frac{a^2 - 1 + 1}{a+1} + \frac{\mathrm{b}^2 - 1 + 1}{\mathrm{b}+1} \overset{?}{\geq} 1$$

$$\Leftrightarrow a - 1 + \frac{1}{a+1} + \mathrm{b} - 1 + \frac{1}{\mathrm{b}+1} \overset{?}{\geq} 1 \Leftrightarrow a + 1 + \frac{1}{a+1} + \mathrm{b} + 1 + \frac{1}{\mathrm{b}+1} \overset{?}{\geq} 5$$

$$\Leftrightarrow x + \frac{1}{x} + \mathrm{y} + \frac{1}{\mathrm{y}} \overset{?}{\geq} 5 \Leftrightarrow (x + \mathrm{y})\left(1 + \frac{1}{x\mathrm{y}}\right) \underset{(*)}{\underbrace{\overset{?}{\geq}}} 5$$

$$\text{Now, } (x+\mathrm{y})\left(1 + \frac{1}{x\mathrm{y}}\right) \overset{\text{A-G}}{\geq} (x+\mathrm{y})\left(1 + \frac{4}{(x+\mathrm{y})^2}\right) \overset{?}{\geq} 5 \Leftrightarrow \mathrm{t} + \frac{4}{\mathrm{t}} \overset{?}{\geq} 5$$

$$\Leftrightarrow \mathrm{t}^2 - 5\mathrm{t} + 4 \overset{?}{\geq} 0 \Leftrightarrow (\mathrm{t}-4)(\mathrm{t}-1) \overset{?}{\geq} 0 \to \text{true} \because \mathrm{t} \geq 4 \text{ via } (1) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a^2}{a+1} + \frac{\mathrm{b}^2}{\mathrm{b}+1} \geq 1 \ \forall\, a, \mathrm{b} > 0 \,\Big|\, a + \mathrm{b} \geq \frac{1}{a^2} + \frac{1}{\mathrm{b}^2}, '' = '' \text{ iff } a = \mathrm{b} = 1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$a + b \geq \frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2}{ab} \geq \frac{8}{(a+b)^2} \quad \Rightarrow \quad a + b \geq 2.$$

By CBS inequality, we get

$$\frac{a^2}{a+1} + \frac{b^2}{b+1} \geq \frac{(a+b)^2}{a+b+2} \overset{a+b\geq 2}{\underbrace{\geq}} \frac{2(a+b)}{a+b+2} \overset{a+b\geq 2}{\underbrace{\geq}} \frac{(a+b)+2}{a+b+2} = 1.$$

Equality holds iff $a = b = 1$.