ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+3$, then prove that :

$$\frac{1}{\sqrt{a^2+ab+2b^2}}+\frac{1}{\sqrt{b^2+bc+2c^2}}+\frac{1}{\sqrt{c^2+ca+2a^2}}\le\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\sqrt{a^2+ab+2b^2}}+\frac{1}{\sqrt{b^2+bc+2c^2}}+\frac{1}{\sqrt{c^2+ca+2a^2}}$$

$$=\sum_{cyc}\frac{1}{\sqrt{(b-c)^2+3bc+c^2}}\le\sum_{cyc}\frac{1}{\sqrt{c(3b+c)}}\overset{CBS}{\le}\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\sum_{cyc}\frac{1}{3b+c}}\overset{A-G}{\le}$$

$$\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\sum_{cyc}\frac{1}{2\sqrt{2b(b+c)}}}\overset{CBS}{\le}\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{2}\sqrt{\sum_{cyc}\frac{1}{2a}}\cdot\sqrt{\sum_{cyc}\frac{1}{b+c}}}\overset{\text{Reverse Bergstrom}}{\le}$$

$$\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{2}\sqrt{\sum_{cyc}\frac{1}{2a}}\cdot\sqrt{\frac{1}{4}\sum_{cyc}\left(\frac{1}{b}+\frac{1}{c}\right)}}=\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{2}\sqrt{\frac{1}{2}\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{2}\sum_{cyc}\frac{1}{a}}}$$

$$=\sqrt{\sum_{cyc}\frac{1}{a}}\cdot\sqrt{\frac{1}{4}\sum_{cyc}\frac{1}{a}}=\frac{1}{2}\sum_{cyc}\frac{1}{a}\overset{CBS}{\le}\frac{\sqrt{3}}{2}\sqrt{\sum_{cyc}\frac{1}{a^2}}$$

$$\therefore\frac{1}{\sqrt{a^2+ab+2b^2}}+\frac{1}{\sqrt{b^2+bc+2c^2}}+\frac{1}{\sqrt{c^2+ca+2a^2}}\overset{(*)}{\le}\frac{\sqrt{3}}{2}\sqrt{\sum_{cyc}\frac{1}{a^2}}$$

$$\text{Now, }3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+3\Rightarrow3\sum_{cyc}\frac{1}{a^2}\le2\sum_{cyc}\frac{1}{a^2}+3$$

$$\Rightarrow\sum_{cyc}\frac{1}{a^2}\overset{(**)}{\le}3\therefore(*),(**)\Rightarrow\frac{1}{\sqrt{a^2+ab+2b^2}}+\frac{1}{\sqrt{b^2+bc+2c^2}}+\frac{1}{\sqrt{c^2+ca+2a^2}}$$

$$\le\frac{3}{2}\ \forall\, a,b,c>0\,\Big|\,3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+3,$$

$$''=''\text{ iff } a=b=c=1\ (QED)$$