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If $a, \mathrm{b}, \mathrm{c} > 0$ and $a^2 + \mathrm{b}^2 + \mathrm{c}^2 = 3$, then prove that :

$$\frac{a^2}{\mathrm{b}} + \frac{\mathrm{b}^2}{\mathrm{c}} + \frac{\mathrm{c}^2}{a} \geq 3$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{\mathrm{b}} + \frac{\mathrm{b}^2}{\mathrm{c}} + \frac{\mathrm{c}^2}{a} = \sum_{\mathrm{cyc}} \frac{a^4}{a^2\mathrm{b}} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\mathrm{cyc}} a^2\right)^2}{\sum_{\mathrm{cyc}} a^2\mathrm{b}} \overset{\text{CBS}}{\geq} \frac{\left(\sum_{\mathrm{cyc}} a^2\right)^2}{\sqrt{\sum_{\mathrm{cyc}} a^2\mathrm{b}^2} \cdot \sqrt{\sum_{\mathrm{cyc}} a^2}}$$

$$\overset{a^2+\mathrm{b}^2+\mathrm{c}^2=3}{=} \frac{\left(\sum_{\mathrm{cyc}} a^2\right)^2}{\sqrt{3\sum_{\mathrm{cyc}} a^2\mathrm{b}^2}} \overset{\left(\sum_{\mathrm{cyc}} a^2\right)^2 \geq 3\sum_{\mathrm{cyc}} a^2\mathrm{b}^2}{\geq} \frac{\left(\sum_{\mathrm{cyc}} a^2\right)^2}{\sum_{\mathrm{cyc}} a^2} = \sum_{\mathrm{cyc}} a^2 = 3$$

$$\therefore \frac{a^2}{\mathrm{b}} + \frac{\mathrm{b}^2}{\mathrm{c}} + \frac{\mathrm{c}^2}{a} \geq 3 \ \forall\ a, \mathrm{b}, \mathrm{c} > 0 \,\middle|\, a^2 + \mathrm{b}^2 + \mathrm{c}^2 = 3, '' ='' \text{ iff } a = \mathrm{b} = \mathrm{c} = 1 \text{ (QED)}$$