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If $x, y > 0$ then:

$$\frac{1}{x}+\frac{1}{y}+\frac{2}{x+y} \geq \frac{3}{\sqrt{xy}}$$

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Multiplying by $\sqrt{xy}$, the inequality is equivalent to

$$\frac{\sqrt{y}}{\sqrt{x}}+\frac{\sqrt{x}}{\sqrt{y}}+\frac{2\sqrt{xy}}{x+y} \geq 0 \Leftrightarrow \frac{y(x+y)+x(x+y)+2xy}{\sqrt{xy}(x+y)} \geq 3 \Leftrightarrow$$

$$\Leftrightarrow (x+y)^2+2xy \geq 3\sqrt{xy}(x+y)$$

Making the substitution $s = x+y$ and $p = \sqrt{xy}$, with $s \geq 2p$ (from AM-GM), we have

$$s^2+2p^2-3ps = s^2-ps+2p^2-2ps = s(s-p)+2p(p-s) = (s-p)(s-2p) \geq 0$$

which is always true.

Equality holds if and only if $x = y$.