ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ then:

$$\frac{1}{1+2a}+\frac{1}{1+2b}\geq\frac{2}{ab+2}$$

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$$\frac{1}{1+2a}+\frac{1}{1+2b}\geq\frac{2}{ab+2}\Leftrightarrow\frac{1+2a+1+2b}{(1+2a)(1+2b)}\geq\frac{2}{ab+2}$$

$$\frac{2(1+a+b)}{1+2(a+b)+4ab}\geq\frac{2}{ab+2},\qquad S=a+b, P=ab$$

$$\frac{1+S}{1+2S+4P}\geq\frac{1}{P+2}\Leftrightarrow(1+S)(P+2)\geq 1+2S+4P$$

$$P+2+SP+2S\geq\ 1+2S+4P,\qquad SP+1\geq 3P\ (to\ prove)$$

$$SP+1\overset{AM-GM}{\overbrace{\geq}}2\sqrt{P}\cdot P+1\geq 3P\ (to\ prove), \sqrt{P}=x\ (denote)$$

$$2x^3+1\geq 3x^2\Leftrightarrow 2x^3+1-3x^2\geq 0$$

$$2x^3-2x^2-x^2+1\geq 0\Leftrightarrow 2x^2(x-1)-(x-1)(x+1)\geq 0$$

$$(x-1)(2x^2-x-1)\geq 0\Leftrightarrow(x-1)^2(2x+1)\geq 0$$

Equality holds for $a=b=1$.