ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0, then:$

$$\sqrt{\frac{a+2b}{a^2+2b^2}}+\sqrt{\frac{b+2a}{b^2+2a^2}} \le \sqrt{\frac{8}{a+b}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{a+2b}{a^2+2b^2}}+\sqrt{\frac{b+2a}{b^2+2a^2}} \overset{\text{CBS}}{\le} \sqrt{a+2b+b+2a}.\sqrt{\frac{1}{a^2+2b^2}+\frac{1}{b^2+2a^2}}$$

$$\overset{?}{\le} \sqrt{\frac{8}{a+b}} \Leftrightarrow 3(a+b).\sqrt{a^2+b^2} \overset{?}{\le} \sqrt{8(a^2+2b^2)(b^2+2a^2)}$$

$$\Leftrightarrow 8\left(a^2+2b^2\right)\left(b^2+2a^2\right) \overset{?}{\ge} 9\left(a^2+b^2\right)(a+b)^2$$

$$\Leftrightarrow 7t^4-18t^3+22t^2-18t+7 \overset{?}{\ge} 0 \left(t=\frac{a}{b}\right) \Leftrightarrow (t-1)^2\left(7(t-1)^2+10t\right) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because t>0 \therefore \sqrt{\frac{a+2b}{a^2+2b^2}}+\sqrt{\frac{b+2a}{b^2+2a^2}} \le \sqrt{\frac{8}{a+b}}, ''='' \text{ iff } a=b \text{ (QED)}$$