ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $a + b = 2$

then : $a^{2b} + b^{2a} \le 2$

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$$a^{2b} = \left(\left(1 + (a-1)\right)^{\frac{b}{2}}\right)^4 \overset{\text{Bernoulli} \because \frac{b}{2} < 1}{\le} \left(1 + \frac{b(a-1)}{2}\right)^4$$

$$= \frac{1}{16}(2 - b + ab)^4 \overset{a+b=2}{=} \frac{1}{16}(a + ab)^4$$

$$= \frac{1}{16}.a^4(1+b)^4 \overset{a+b=2}{=} \frac{1}{16}.(2-b)^4(1+b)^4$$

$$\therefore a^{2b} \le \frac{1}{16}.(2-b)^4(1+b)^4 \to (1)$$

$$\text{Again}, b^{2a} = \left(\left(1 + (b-1)\right)^{\frac{a}{2}}\right)^4 \overset{\text{Bernoulli} \because \frac{a}{2} < 1}{\le} \left(1 + \frac{a(b-1)}{2}\right)^4$$

$$= \frac{1}{16}(2 - a + ab)^4 \overset{a+b=2}{=} \frac{1}{16}(b + ab)^4 = \frac{1}{16}.b^4(1+a)^4 \overset{a+b=2}{=} \frac{1}{16}.b^4(3-b)^4$$

$$\therefore b^{2a} \le \frac{1}{16}.b^4(3-b)^4 \to (2) \therefore (1),(2) \Rightarrow a^{2b} + b^{2a}$$

$$\le \frac{1}{16}.(2-b)^4(1+b)^4 + \frac{1}{16}.b^4(3-b)^4 \overset{?}{\le} 2$$

$$\Leftrightarrow b^8 - 8b^7 + 26b^6 - 44b^5 + 41b^4 - 20b^3 - 4b^2 + 16b - 8 \underbrace{\overset{?}{\le}}_{(*)} 0$$

$$\Leftrightarrow (b-1)^2\left(b^6 - 6b^5 + 13b^4 - 12b^3 + 4b^2 - 8\right) \overset{?}{\le} 0 \text{ and to prove this,}$$

$$\text{it suffices to prove : } b^6 - 6b^5 + 13b^4 - 12b^3 + 4b^2 < 8$$

$$\Leftrightarrow \boxed{\boxed{b^2(b-1)^2(b-2)^2 \overset{(**)}{<} 8}}$$

$$\text{Now}, \because 0 < b < 2 \therefore -1 < b - 1 < 1 \Rightarrow |b-1| < 1 \Rightarrow \boxed{(b-1)^2 < 1} \to \text{(i)}$$

$$\text{Also}, 2 = a + b \overset{A-G}{\ge} 2\sqrt{ab} \Rightarrow ab \le 1 \Rightarrow b^2(b-2)^2 = b^2(2-b)^2 \overset{a+b=2}{=} a^2b^2 \le 1$$

$$\Rightarrow \boxed{b^2(b-2)^2 \le 1} \to \text{(ii)}$$

$$\therefore \text{(i)}\bullet\text{(ii)} \Rightarrow b^2(b-1)^2(b-2)^2 < 1 < 8 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore a^{2b} + b^{2a} \le 2 \ \forall \ a, b > 0 \,\middle|\, a + b = 2, ''='' \text{ iff } a = b = 1 \text{ (QED)}$$