ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b} > 0$ and $a + \mathrm{b} = a^4 + \mathrm{b}^4$

then : $a^a \mathrm{b}^\mathrm{b} \leq 1$

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$$\text{Via weighted AM} - \text{GM inequality}, \sqrt[a+\mathrm{b}]{(a^3)^a.(\mathrm{b}^3)^\mathrm{b}} \leq \frac{a^3.a + \mathrm{b}^3.\mathrm{b}}{a+\mathrm{b}}$$

$$= \frac{a^4 + \mathrm{b}^4}{a + \mathrm{b}} = 1 \Rightarrow \frac{1}{a+\mathrm{b}}.\ln x \leq 0 \ \left(x = \left(a^3\right)^a.\left(\mathrm{b}^3\right)^\mathrm{b}\right) \Rightarrow \ln x \leq 0$$

$$\Rightarrow x = \left(a^3\right)^a.\left(\mathrm{b}^3\right)^\mathrm{b} \leq 1 \Rightarrow (a^a)^3.\left(\mathrm{b}^\mathrm{b}\right)^3 \leq 1$$

$$\Rightarrow \left(a^a\mathrm{b}^\mathrm{b}\right)^3 \leq 1 \Rightarrow a^a\mathrm{b}^\mathrm{b} \leq 1 \ \forall\, a, \mathrm{b} > 0 \,\Big|\, a + \mathrm{b} = a^4 + \mathrm{b}^4, '' ='' \text{ iff } a = \mathrm{b} = 1 \text{ (QED)}$$