ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } a, b > 0 \text{ and } a^9 + b^9 = 2$$

$$\text{then : } \frac{a^2}{b} + \frac{b^2}{a} \ge 2$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$a^9 + b^9 = 2 \Rightarrow (a^3 + b^3)(a^6 + b^6 - a^3b^3) = 2$$

$$\Rightarrow (a^3 + b^3)\left((a^3 + b^3)^2 - 3a^3b^3\right) = 2 \Rightarrow (a^3 + b^3)^3 - 2 = 3a^3b^3(a^3 + b^3)$$

$$\Rightarrow a^3b^3 = \frac{(a^3 + b^3)^3 - 2}{3(a^3 + b^3)} \to (1)$$

$$\text{Now,} \frac{a^2}{b} + \frac{b^2}{a} \ge 2 \Leftrightarrow a^3 + b^3 \ge 2ab \Leftrightarrow \frac{(a^3 + b^3)^3}{8} \ge a^3b^3$$

$$\overset{\text{via (1)}}{\Leftrightarrow} \frac{(a^3 + b^3)^3}{8} \ge \frac{(a^3 + b^3)^3 - 2}{3(a^3 + b^3)} \Leftrightarrow 3t^4 \ge 8(t^3 - 2) \ (t = a^3 + b^3)$$

$$\Leftrightarrow 3t^4 - 8t^3 + 16 \ge 0 \Leftrightarrow (t - 2)^2(3t^2 + 4t + 4) \ge 0 \to \text{true}$$

$$\because t > 0 \therefore \frac{a^2}{b} + \frac{b^2}{a} \ge 2 \ \forall \ a, b > 0 \mid a^9 + b^9 = 2, '' ='' \text{ iff } a = b = 1 \ (\text{QED})$$