ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a^2 + b^2 + c^2 = a + b + c$, then :
$a^2b^2 + b^2c^2 + c^2a^2 \leq ab + bc + ca$

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Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$$\therefore abc = r^2 s \rightarrow (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4),$$

$$\text{and } \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R + r)^2 - 2s^2\right) \rightarrow (5)$$

$$a^2b^2 + b^2c^2 + c^2a^2 \leq ab + bc + ca \overset{a^2+b^2+c^2 = a+b+c}{\Leftrightarrow}$$

$$\sum_{cyc} a^2b^2 \leq \left(\sum_{cyc} ab\right)\left(\frac{\sum_{cyc} a^2}{\sum_{cyc} a}\right)^2 \overset{\text{via (1),(3),(4) and (5)}}{\Leftrightarrow} (4Rr + r^2)(s^2 - 8Rr - 2r^2)^2$$

$$\geq s^2r^2\left((4R + r)^2 - 2s^2\right) \Leftrightarrow (4R + 3r)s^4 - 5rs^2(4R + r)^2 + 4r^2(4R + r)^3 \overset{(*)}{\geq} 0$$

$$\text{and } \because (4R + 3r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \geq (4R + 3r)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (48R^2 + 16Rr - 35r^2)s^2 \overset{(**)}{\geq} r(768R^3 - 64R^2r - 428Rr^2 + 71r^3)$$

$$\text{Now, } (48R^2 + 16Rr - 35r^2)s^2 \overset{\text{Gerretsen}}{\geq} (48R^2 + 16Rr - 35r^2)(16Rr - 5r^2)$$

$$\overset{?}{\geq} r(768R^3 - 64R^2r - 428Rr^2 + 71r^3) \Leftrightarrow 20R^2 - 53Rr + 26r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R - 2r)(20R - 13r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore a^2b^2 + b^2c^2 + c^2a^2 \leq ab + bc + ca$$

$$\forall a, b, c > 0 \,\middle|\, a^2 + b^2 + c^2 = a + b + c, '' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$