ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc(a+b+c)^3 = 27$, then prove that :

$$(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq 8$$

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$$(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq \prod_{cyc}\left(\frac{1}{2}(a+b)^2\right) \overset{?}{\geq} 8 \Leftrightarrow \prod_{cyc}(a+b)^2 \overset{?}{\geq} 64$$

$$= \frac{64}{27}.27 \overset{abc(a+b+c)^3=27}{=} \frac{64}{27}.abc(a+b+c)^3 \Leftrightarrow 27\prod_{cyc}(a+b)^2 \underbrace{\overset{?}{\geq}}_{(*)} 64abc\left(\sum_{cyc} a\right)^3$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a > 0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$$\therefore abc = \prod_{cyc}(s-x) \Rightarrow abc = r^2 s \rightarrow (2) \therefore (1),(2) \Rightarrow (*) \Leftrightarrow 27x^2y^2z^2 \geq 64r^2s.s^3$$

$$\Leftrightarrow 27.16R^2r^2s^2 \geq 64r^2s^4 \Leftrightarrow 27R^2 \geq 4s^2 \Leftrightarrow s \leq \frac{3\sqrt{3}R}{2} \rightarrow \text{true via Mitrinovic}$$

$\Rightarrow (*)$ is true $\therefore (a^2+b^2)(b^2+c^2)(c^2+a^2) \geq 8$

$\forall\, a, b, c > 0 \,\big|\, abc(a+b+c)^3 = 27,$ " $=$ " iff $a = b = c = 1$ (QED)