ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $(a+b)(b+c)(c+a) = 1$, then prove that :

$$\frac{a}{b(b+2c)^2} + \frac{b}{c(c+2a)^2} + \frac{c}{a(a+2b)^2} \geq \frac{4}{3}$$

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$$\frac{a}{b(b+2c)^2} + \frac{b}{c(c+2a)^2} + \frac{c}{a(a+2b)^2} = \frac{a^2}{ab(b+2c)^2} + \frac{b^2}{bc(c+2a)^2} + \frac{c^2}{ca(a+2b)^2}$$

$$= \frac{\left(\frac{a}{b+2c}\right)^2}{ab} + \frac{\left(\frac{b}{c+2a}\right)^2}{bc} + \frac{\left(\frac{c}{a+2b}\right)^2}{ca} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\text{cyc}} \frac{a^2}{ab+2ca}\right)^2}{\sum_{\text{cyc}} ab} \overset{\text{Bergstrom}}{\geq} \frac{\left(\frac{\left(\sum_{\text{cyc}} a\right)^2}{3\sum_{\text{cyc}} ab}\right)^2}{\sum_{\text{cyc}} ab}$$

$$\overset{\left(\sum_{\text{cyc}} a\right)^2 \geq 3\sum_{\text{cyc}} ab}{\geq} \frac{\left(\sum_{\text{cyc}} a\right)^2}{3\left(\sum_{\text{cyc}} ab\right)^2} \overset{?}{\geq} \frac{4}{3} \Leftrightarrow \sum_{\text{cyc}} a \overset{?}{\geq} 2\sum_{\text{cyc}} ab$$

$$\overset{(a+b)(b+c)(c+a)=1}{\Leftrightarrow} \left(\sum_{\text{cyc}} a\right) \cdot \sqrt[3]{(a+b)(b+c)(c+a)} \underset{(*)}{\overset{?}{\geq}} 2\sum_{\text{cyc}} ab$$

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{\text{cyc}} a = \sum_{\text{cyc}} x = 2s \Rightarrow \sum_{\text{cyc}} a = s \rightarrow (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$$\therefore \sum_{\text{cyc}} ab = \sum_{\text{cyc}} (s-x)(s-y) = 4Rr + r^2 \rightarrow (2) \therefore \rightarrow (1), (2) \Rightarrow (*) \Leftrightarrow$$

$$s.\sqrt[3]{xyz} \geq 2(4Rr + r^2) \Leftrightarrow s^3.4Rs \geq 8r^3(4R+r)^3 \Leftrightarrow Rs^4 \overset{(**)}{\geq} 2r^2(4R+r)^3$$

$$\text{Now, } s^2 \overset{\text{Gerretsen}}{\geq} 16Rr - 5r^2 = 3r(4R+r) + 4r(R-2r) \overset{\text{Euler}}{\geq} 3r(4R+r)$$

$\therefore$ in order to prove $(**)$, it suffices to prove : $R\left(3r(4R+r)\right)^2 \geq 2r^2(4R+r)^3$

$\Leftrightarrow 9R \geq 2(4R+r) \Leftrightarrow R \geq 2r \rightarrow$ true via Euler $\Rightarrow (**) \Rightarrow (*)$ is true

$$\therefore \frac{a}{b(b+2c)^2} + \frac{b}{c(c+2a)^2} + \frac{c}{a(a+2b)^2} \geq \frac{4}{3}$$

$$\forall\, a, b, c > 0 \,\middle|\, (a+b)(b+c)(c+a) = 1, \text{''} = \text{''} \text{ iff } a = b = c = \frac{1}{2} \text{ (QED)}$$