ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{a}{\sqrt{b^2+2c}}+\frac{b}{\sqrt{c^2+2a}}+\frac{c}{\sqrt{a^2+2b}} \geq \sqrt{3}$$

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$$\frac{a}{\sqrt{b^2+2c}}+\frac{b}{\sqrt{c^2+2a}}+\frac{c}{\sqrt{a^2+2b}}$$

$$= \frac{a\sqrt{a}}{\sqrt{b^2a+2ca}}+\frac{b\sqrt{b}}{\sqrt{c^2b+2ab}}+\frac{c\sqrt{c}}{\sqrt{a^2c+2bc}} \overset{\text{Radon}}{\geq} \frac{\left(\sum_{cyc} a\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc} ab^2 + 2\sum_{cyc} ab}} \overset{?}{\geq} \sqrt{3}$$

$$\Leftrightarrow \left(\sum_{cyc} a\right)^3 \overset{?}{\geq} 3\sum_{cyc} ab^2 + 6\sum_{cyc} ab$$

$$\Leftrightarrow \sum_{cyc} a^3 + 6abc + 3\sum_{cyc} a^2b + 3\sum_{cyc} ab^2 \overset{?}{\geq} 3\sum_{cyc} ab^2 + 6\sum_{cyc} ab$$

$$\Leftrightarrow \sum_{cyc} a^3 + 6abc + 3\sum_{cyc} a^2b \boxed{\overset{?}{\underset{(\bullet)}{\geq}}} 6\sum_{cyc} ab$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc} (s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3), \sum_{cyc} a^3 = \left(\sum_{cyc} a\right)^3 - 3(a+b)(b+c)(c+a)$$

$$\overset{\text{via }(1)}{=} s^3 - 3xyz \Rightarrow \sum_{cyc} a^3 = s^3 - 12Rrs \rightarrow (4)$$

Now, via Bergstrom and via (2), (3) and (4), LHS of $(\bullet)$ − RHS of $(\bullet)$

$$\geq s^3 - 12Rrs + 6r^2s - 6(4Rr + r^2) + \frac{3\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} a} \overset{\text{via }(1)\text{ and }(3)}{=}$$

$$\frac{s^2(s^2 - 12Rr + 6r^2) - 6s(4Rr + r^2) + 3(4Rr + r^2)^2}{s} \overset{?}{\geq} 0$$

$$\Leftrightarrow s^2(s^2 - 12Rr + 6r^2) + 3(4Rr + r^2)^2 \overset{?}{\geq} 6s(4Rr + r^2)$$

$$\overset{abc=1}{=} 6s(4Rr + r^2).\sqrt[3]{abc} \overset{\text{via }(2)}{=} 6s(4Rr + r^2).\sqrt[3]{r^2 s}$$

$$\Leftrightarrow \left(s^2(s^2-12Rr+6r^2)+3(4Rr+r^2)^2\right)^3 \overset{?}{\geq} 216r^2s^4(4Rr+r^2)^3$$

$$\Leftrightarrow \left(s^2(s^2-12Rr+6r^2)+3(4Rr+r^2)^2\right)^3 - 216r^2s^4(4Rr+r^2)^3 \boxed{\overset{?}{\underset{(*)}{\geq}}} 0 \text{ and}$$

$$\because \text{via Gerretsen}, P = (s^2-16Rr+5r^2)^6 + (60Rr-12r^2)(s^2-16Rr+5r^2)^5$$
$$+2r^2(768R^2-210Rr+21r^2)(s^2-16Rr+5r^2)^4$$
$$+4r^3(5360R^3-1272R^2r+60Rr^2-4r^3)(s^2-16Rr+5r^2)^3$$
$$+4r^4(43008R^4-7536R^2r-4284R^2r^2-228Rr^3-75r^4)(s^2-16Rr+5r^2)^2$$
$$+16r^5\begin{pmatrix}47040R^5-20256R^4r-15636R^3r^2+1032R^2r^3\\+1263Rr^4+132r^5\end{pmatrix}(s^2-16Rr+5r^2) \geq 0$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \geq P \Leftrightarrow$$

$$87808t^6-174144t^5-23952t^4+39428t^3+6198t^2-1875t-338 \geq 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}87808t^4+177088t^3+333168t^2\\+663748t+1328518\end{pmatrix}+2657205\right) \geq 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \Rightarrow (\bullet) \text{ is true} \therefore \frac{a}{\sqrt{b^2+2c}}+\frac{b}{\sqrt{c^2+2a}}+\frac{c}{\sqrt{a^2+2b}} \geq \sqrt{3}$$

$$\forall a,b,c>0 \,\middle|\, abc=1, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$