ROMANIAN MATHEMATICAL MAGAZINE

If $a+b+c=3$ and $a,b,c\in\mathbb{R}$, then :

$$\frac{a^2-bc}{a^2+3}+\frac{b^2-ca}{b^2+3}+\frac{c^2-ab}{c^2+3}\geq 0$$

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$$\frac{a^2-bc}{a^2+3}+\frac{b^2-ca}{b^2+3}+\frac{c^2-ab}{c^2+3}\geq 0 \overset{a+b+c=3}{\Leftrightarrow}$$
$$\frac{a^2-bc}{a^2+\frac{(\sum_{cyc}a)^2}{3}}+\frac{b^2-ca}{b^2+\frac{(\sum_{cyc}a)^2}{3}}+\frac{c^2-ab}{c^2+\frac{(\sum_{cyc}a)^2}{3}}\geq 0$$

$$\Leftrightarrow \sum_{cyc}\left((a^2-bc)\left(3b^2+\left(\sum_{cyc}a\right)^2\right)\left(3c^2+\left(\sum_{cyc}a\right)^2\right)\right)\geq 0$$

$$\Leftrightarrow \sum_{cyc}a^6+27a^2b^2c^2+3\sum_{cyc}a^4b^2+3\sum_{cyc}a^2b^4 \overset{(\bullet)}{\geq} 9abc\sum_{cyc}a^3+\sum_{cyc}a^3b^3$$
$$+3abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)$$

$$\because x^2+y^2+z^2-xy-yz-zx=\frac{1}{2}\sum_{cyc}(x-y)^2\geq 0\ \forall\, x,y,z\in\mathbb{R}$$

$$\therefore 3\sum_{cyc}a^4b^2+3\sum_{cyc}a^2b^4\geq 3abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\ \forall\, a,b,c\in\mathbb{R}\rightarrow \text{(i)}$$

Case 1 Exactly 2 variables ≤ 0 and WLOG we may assume $b,c\leq 0$ and then :

$$a=3-b-c\geq 3\ \therefore -ca\geq 0 \text{ and } -ab\geq 0\ \therefore \frac{a^2-bc}{a^2+3}+\frac{b^2-ca}{b^2+3}+\frac{c^2-ab}{c^2+3}$$
$$\geq \frac{a^2}{a^2+3}-\frac{bc}{a^2+3}\overset{A-G}{\geq}\frac{a^2}{a^2+3}-\frac{(b+c)^2}{4(a^2+3)}\overset{a+b+c=3}{=}\frac{4a^2-(3-a)^2}{4(a^2+3)}$$
$$=\frac{3(a^2+2a-3)}{4(a^2+3)}=\frac{3(a-1)(a+3)}{4(a^2+3)}>0\ (\because a\geq 3)$$
$$\therefore \frac{a^2-bc}{a^2+3}+\frac{b^2-ca}{b^2+3}+\frac{c^2-ab}{c^2+3}>0$$

Case 2 Exactly 1 variable ≤ 0 and WLOG we may assume $a\leq 0$ $(b,c\geq 0)$

$$\text{and then : } 9abc\sum_{cyc}a^3-27a^2b^2c^2$$

$$= 9abc\left(3abc + \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)\right) - 27a^2b^2c^2 \overset{a+b+c=3}{=}$$

$$\frac{27}{2}abc\sum_{cyc}(a-b)^2 \leq 0 \;(\because a \leq 0 \text{ and } b, c \geq 0) \Rightarrow 27a^2b^2c^2 \geq 9abc\sum_{cyc} a^3 \to (ii)$$

and via A − G, $\sum_{cyc} a^6 > \sum_{cyc} a^3b^3 \to (iii)$ (strict inequality for $a \leq 0$ and $b, c \geq 0$)

$$\therefore (i) + (ii) + (iii) \Rightarrow (\bullet) \text{ is true } \therefore \frac{a^2 - bc}{a^2 + 3} + \frac{b^2 - ca}{b^2 + 3} + \frac{c^2 - ab}{c^2 + 3} > 0$$

Case 3 $a, b, c > 0$ and assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$ $\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x,$

$b = s - y, c = s - z \therefore abc = r^2s \to (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab =$

$$\sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3),$$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4),$$

$$\sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R + r)^2 - 2s^2\right) \to (5)$$

$$\text{Now, } (\bullet) \Leftrightarrow 3a^2b^2c^2 + \left(\sum_{cyc} a^2\right)\left(\left(\sum_{cyc} a^2\right)^2 - 3\sum_{cyc} a^2b^2\right) + 27a^2b^2c^2$$

$$+3\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - 9a^2b^2c^2$$

$$\geq 9abc\left(3abc + \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)\right) + 3a^2b^2c^2$$

$$+\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2 - abc\sum_{cyc} a\right) + 3abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9a^2b^2c^2$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^3\geq 9abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$+\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2-abc\sum_{cyc}a\right)+3abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)$$

$$\overset{\text{via }(1),(2),(3),(4)\text{ and }(5)}{\Leftrightarrow}\left(s^2-8Rr-2r^2\right)^3\geq 9r^2s^2\left(s^2-8Rr-2r^2-4Rr-r^2\right)$$

$$+\left(4Rr+r^2\right)\left(r^2\left((4R+r)^2-2s^2\right)-r^2s^2\right)+3r^2s^2\left(4Rr+r^2\right)$$

$$\Leftrightarrow s^6-\left(24Rr+15r^2\right)s^4+r^2s^2\left(192R^2+204Rr+39r^2\right)-9r^3(4R+r)^3\overset{(*)}{\geq}0$$

and $\because\left(s^2-16Rr+5r^2\right)^3\overset{\text{Gerretsen}}{\geq}0\therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of }(*)\geq\left(s^2-16Rr+5r^2\right)^3$$

$$\Leftrightarrow(12R-15r)s^4-rs^2\left(288R^2-342Rr+18r^2\right)$$

$$+r^2\left(1760R^3-2136R^2r+546Rr^2-67r^3\right)\overset{(**)}{\geq}0\text{ and}$$

$\because(12R-15r)\left(s^2-16Rr+5r^2\right)^2\overset{\text{Gerretsen}}{\geq}0\therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**)\geq(12R-15r)\left(s^2-16Rr+5r^2\right)^2$

$$\Leftrightarrow\left(48R^2-129Rr+66r^2\right)s^2\geq r\left(656R^3-1812R^2r+1077Rr^2-154r^3\right)$$

$$\Leftrightarrow(R-2r)(48R-33r)s^2\geq r(R-2r)\left(656R^2-500Rr+77r^2\right)$$

$$\Leftrightarrow(48R-33r)s^2\overset{(***)}{\geq}r\left(656R^2-500Rr+77r^2\right)\left(\because R-2r\overset{\text{Euler}}{\geq}0\right)$$

Now, $(48R-33r)s^2\overset{\text{Gerretsen}}{\geq}(48R-33r)\left(16Rr-5r^2\right)\overset{?}{\geq}$

$$r\left(656R^2-500Rr+77r^2\right)\Leftrightarrow 28R^2-67Rr+22r^2\overset{?}{\geq}0$$

$$\Leftrightarrow(R-2r)(28R-11r)\overset{?}{\geq}0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\Rightarrow(\bullet)$$

is true $\therefore\dfrac{a^2-bc}{a^2+3}+\dfrac{b^2-ca}{b^2+3}+\dfrac{c^2-ab}{c^2+3}\geq 0\ \forall\ a,b,c\in\mathbb{R}\,\middle|\,a+b+c=3$,

" = " iff $a=b=c=1$ (QED)