ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, then prove that :

$$(xy + yz + zx)\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right)^2 \ge 27$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \sqrt{x} = a, \sqrt{y} = b, \sqrt{z} = c \text{ and then : } x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\Rightarrow \sum_{cyc} a^2 = \frac{\sum_{cyc} a^2b^2}{a^2b^2c^2} \Rightarrow 1 = \frac{a^2b^2c^2\sum_{cyc} a^2}{\sum_{cyc} a^2b^2} \to (i)$$

Assigning $b + c = X, c + a = Y, a + b = Z \Rightarrow X + Y - Z = 2c > 0, Y + Z - X = 2a > 0$ and $Z + X - Y = 2b > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - X, b = s - Y, c = s - Z$

$\therefore abc = r^2s \to (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s - X)(s - Y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4),$$

$$\sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R + r)^2 - 2s^2\right) \to (5)$$

$$\text{Now, } (xy + yz + zx)\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right)^2 \ge 27 \Leftrightarrow \left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} ab\right)^2 \ge 27$$

$$\overset{\text{via (i)}}{=} 27\left(\frac{a^2b^2c^2\sum_{cyc} a^2}{\sum_{cyc} a^2b^2}\right)^2 \Leftrightarrow \left(\sum_{cyc} a^2b^2\right)^3\left(\sum_{cyc} ab\right)^2 \ge 27(abc)^4\left(\sum_{cyc} a^2\right)^2$$

$$\overset{\text{via (2),(3),(4) and (5)}}{\Leftrightarrow} r^6\left((4R + r)^2 - 2s^2\right)^3 . r^2(4R + r)^2 \ge 27r^8s^4\left(s^2 - 8Rr - 2r^2\right)^2$$

$$\Leftrightarrow \left((4R + r)^2 - 2s^2\right)^3(4R + r)^2 \boxed{\overset{(*)}{\ge}} 27s^4\left(s^2 - 8Rr - 2r^2\right)^2$$

$$\text{Now, LHS of } (*) - \text{RHS of } (*) \overset{\text{Gerretsen}}{\ge}$$

$$\left((4R+r)^2-2\left(4R^2+4Rr+3r^2\right)\right)^3(4R+r)^2$$

$$-27\left(4R^2+4Rr+3r^2\right)^2\left(4R^2+4Rr+3r^2-8Rr-2r^2\right)^2\overset{?}{\geq}0$$

$$\Leftrightarrow 160t^8+512t^7-1856t^6-96t^5+756t^4+600t^3-391t^2+37t-46\overset{?}{\geq}0$$

$$\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(\begin{matrix}160t^6+1152t^5+2112t^4+3744t^3\\+7284t^2+14760t+29513\end{matrix}\right)+59049\right)\overset{?}{\geq}0$$

$$\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true}$$

$$\therefore (xy+yz+zx)\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)^2\geq 27$$

$$\forall x,y,z>0\,\middle|\,x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z},''='' \text{ iff } x=y=z=1\ (\text{QED})$$