ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca = 1$, then prove that :

$$\frac{1 + a^2b^2}{(a + b)^2} + \frac{1 + b^2c^2}{(b + c)^2} + \frac{1 + c^2a^2}{(c + a)^2} \geq \frac{5}{2}$$

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Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s - x)(s - y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4)$ and we have :

$$\frac{1 + a^2b^2}{(a + b)^2} + \frac{1 + b^2c^2}{(b + c)^2} + \frac{1 + c^2a^2}{(c + a)^2} \overset{ab+bc+ca=1}{=} \sum_{cyc} \frac{(ab + bc + ca)^2 + a^2b^2}{(a + b)^2}$$

$$= \sum_{cyc} \frac{2a^2b^2 + c^2(a + b)^2 + 2abc(a + b)}{(a + b)^2}$$

$$= 2\left(\sum_{cyc} \frac{ab}{a + b}\right)^2 - 4abc\sum_{cyc} \frac{a}{(a + b)(c + a)} + \sum_{cyc} a^2 + 2abc\sum_{cyc} \frac{1}{a + b}$$

$$= 2\left(\frac{1}{\prod_{cyc}(a + b)} \cdot \sum_{cyc}(ab(b + c)(c + a))\right)^2 - \frac{4abc}{\prod_{cyc}(a + b)} \cdot \sum_{cyc} a(b + c) + \sum_{cyc} a^2$$
$$+ \frac{2abc}{\prod_{cyc}(a + b)}\sum_{cyc}(b + c)(c + a)$$

$$= \frac{2}{\prod_{cyc}(a + b)^2}\left(\left(\sum_{cyc} ab\right)^2 + abc\left(\sum_{cyc} a\right)\right)^2 - \frac{8abc}{\prod_{cyc}(a + b)}\left(\sum_{cyc} ab\right) + \sum_{cyc} a^2$$
$$+ \frac{2abc}{\prod_{cyc}(a + b)} \cdot \left(\left(\sum_{cyc} a\right)^2 + \sum_{cyc} ab\right) \overset{\text{via (1),(2),(3) and (4)}}{=}$$

$$\frac{2r^4\left((4R+r)^2+s^2\right)^2}{16R^2r^2s^2}+s^2-8Rr-2r^2-\frac{8r^2sr(4R+r)}{4Rrs}+\frac{2r^2s}{4Rrs}\left(s^2+4Rr+r^2\right)$$

$$=\frac{r^2\left((4R+r)^2+s^2\right)^2}{8R^2s^2}+s^2-8Rr-2r^2+r.\frac{s^2+4Rr+r^2-16Rr-4r^2}{2R}$$

$$=\frac{8R^2s^2\left(s^2-8Rr-2r^2\right)+r^2\left((4R+r)^2+s^2\right)^2+4Rrs^2\left(s^2-12Rr-3r^2\right)}{8R^2s^2}\overset{?}{\geq}\frac{5}{2}$$

$$\overset{ab+bc+ca=1}{=}\frac{5}{2}\left(\sum_{cyc}ab\right)\overset{via\,(3)}{=}\frac{5\left(4Rr+r^2\right)}{2}$$

$$\Leftrightarrow\left(8R^2+4Rr+r^2\right)s^4-rs^2\left(144R^3+52R^2r-4Rr^2-2r^3\right)+r^2(4R+r)^4\overset{?}{\underset{(*)}{\geq}}0$$

$and\because\left(8R^2+4Rr+r^2\right)\left(s^2-16Rr+5r^2\right)^2\overset{\text{Gerretsen}}{\geq}0\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\geq\left(8R^2+4Rr+r^2\right)\left(s^2-16Rr+5r^2\right)^2$

$$\Leftrightarrow\left(28R^3-R^2r-Rr^2-2r^3\right)s^2\overset{(**)}{\geq}r\left(448R^4-128R^3r-70R^2r^2-19Rr^3+6r^4\right)$$

Now, $\left(28R^3-R^2r-Rr^2-2r^3\right)s^2\overset{\text{Rouche}}{\geq}$

$$\left(28R^3-R^2r-Rr^2-2r^3\right)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\geq}r\left(448R^4-128R^3r-70R^2r^2-19Rr^3+6r^4\right)$$

$$\Leftrightarrow(R-2r)\left(56R^4-58R^3r-28R^2r^2+Rr^3+2r^4\right)\overset{?}{\underset{(***)}{\geq}}$$

$$2(R-2r)\left(28R^3-R^2r-Rr^2-2r^3\right).\sqrt{R^2-2Rr}$$

$$\because 56R^4-58R^3r-28R^2r^2+Rr^3+2r^4$$

$$=(R-2r)\left(56R^3+54R^2r+80Rr^2+161r^3\right)+324r^4\overset{\text{Euler}}{\geq}324r^4>0\ and$$

$\because R-2r\overset{\text{Euler}}{\geq}0\therefore$ in order to prove $(***)$, it suffices to prove :

$$\left(56R^4-58R^3r-28R^2r^2+Rr^3+2r^4\right)^2>4\left(R^2-2Rr\right)\left(28R^3-R^2r-Rr^2-2r^3\right)^2$$

$$\Leftrightarrow 3360t^5-8t^4-264t^3-95t^2+36t+4>0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(3360t^4+6712t^3+13160t^2+26225t+52486\right)+104976>0$$

$\rightarrow$ true $\because t\overset{\text{Euler}}{\geq}2\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)$ is true $\therefore\frac{1+a^2b^2}{(a+b)^2}+\frac{1+b^2c^2}{(b+c)^2}+\frac{1+c^2a^2}{(c+a)^2}$

$$\geq\frac{5}{2}\ \forall\, a,b,c>0\ \middle|\ ab+bc+ca=1,''='' \text{ iff } a=b=c=\frac{1}{\sqrt{3}}\ \text{(QED)}$$