ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 3$, then prove that :
$$(a + 2b)(b + 2c)(c + 2a) \geq 27$$

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Let $\sqrt{a} = A, \sqrt{b} = B, \sqrt{c} = C$ and then : assigning $B + C = x, C + A = y$,
$A + B = z \Rightarrow x + y - z = 2C > 0, y + z - x = 2A > 0$ and $z + x - y = 2B > 0$
$\Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with
semiperimeter, circumradius and inradius $= s, R, r$ (say) and $2\sum_{cyc} A = \sum_{cyc} x = 2s$

$$\Rightarrow \sum_{cyc} A = s \to (1) \Rightarrow A = s - x, B = s - y, C = s - z \therefore ABC = r^2 s \to (2)$$

and such substitutions $\Rightarrow \sum_{cyc} AB = \sum_{cyc}(s - x)(s - y) \Rightarrow \sum_{cyc} AB = 4Rr + r^2 \to (3)$,

$$\sum_{cyc} A^2 = \left(\sum_{cyc} A\right)^2 - 2\sum_{cyc} AB \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} A^2$$

$$= s^2 - 8Rr - 2r^2 \to (4) \text{ and also}, \sum_{cyc} A^2B^2 = \left(\sum_{cyc} AB\right)^2 - 2ABC\left(\sum_{cyc} A\right)$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s^2 \Rightarrow \sum_{cyc} A^2B^2 = r^2((4R + r)^2 - 2s^2) \to (5)$$

$$\text{Now}, (a + 2b)(b + 2c)(c + 2a) = (A^2 + 2B^2)(B^2 + 2C^2)(C^2 + 2A^2)$$
$$= 2\sum_{cyc} A^4B^2 + 2\sum_{cyc} A^2B^4 + 9A^2B^2C^2 + 2\sum_{cyc} A^2B^4$$
$$= 2\sum_{cyc}\left(A^2B^2\left(\sum_{cyc} A^2 - C^2\right)\right) + 9A^2B^2C^2 + 2\sum_{cyc} A^2B^4$$

$$\overset{A-G}{\geq} 2\left(\sum_{cyc} A^2\right)\left(\sum_{cyc} A^2B^2\right) + 3A^2B^2C^2 + 6A^2B^2C^2 \overset{?}{\geq} 27 \overset{\sum_{cyc} AB = 3}{=} \left(\sum_{cyc} AB\right)^3$$

$$\overset{\text{via (2),(3),(4) and (5)}}{\Leftrightarrow} 2(s^2 - 8Rr - 2r^2).r^2((4R + r)^2 - 2s^2) + 9r^4s^2 \overset{?}{\geq} (4Rr + r^2)^3$$

$$\Leftrightarrow 4s^4 - (32R^2 + 48Rr + 19r^2)s^2 + 5r(4R + r)^3 \boxed{\overset{(*)}{\leq}} 0$$

Now, Rouche $\Rightarrow s^2 - (m - n) \geq 0$ and $s^2 - (m + n) \leq 0$,
where $m = 2R^2 + 10Rr - r^2$ and $n = 2(R - 2r).\sqrt{R^2 - 2Rr}$
$$\therefore \left(s^2 - (m + n)\right)\left(s^2 - (m - n)\right) \leq 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0$$

$$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \leq 0 \therefore \text{in order to prove } (*),$$
$$\text{it suffices to prove : LHS of } (*) \leq 4s^4 - 4s^2(4R^2 + 20Rr - 2r^2) + 4r(4R + r)^3$$
$$\Leftrightarrow (16R^2 - 32Rr + 27r^2)s^2 \boxed{\overset{(**)}{\geq}} r(4R + r)^3$$
$$\text{Again}, (16R^2 - 32Rr + 27r^2)s^2 \overset{\text{Gerretsen}}{\geq} (16R^2 - 32Rr + 27r^2)(16Rr - 5r^2)$$
$$\overset{?}{\geq} r(4R + r)^3 \Leftrightarrow 48t^3 - 160t^2 + 145t - 34 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t - 2)(32t(t - 2) + 16t^2 + 17) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore (a + 2b)(b + 2c)(c + 2a) \geq 27 \ \forall\, a, b, c > 0 \,\Big|\, \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 3,$$
$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$