ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $2(a+b+c)+ab+bc+ca=9$, then prove that :

$$\frac{a+1}{a^2+10a+21}+\frac{b+1}{b^2+10b+21}+\frac{c+1}{c^2+10c+21}\le\frac{3}{16}$$

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$$\sum_{cyc}\frac{a+1}{a^2+10a+21}=\sum_{cyc}\frac{3(a+3)-(a+7)}{2(a+3)(a+7)}\le\frac{3}{16}$$

$$\Leftrightarrow\sum_{cyc}\frac{3}{a+7}-\sum_{cyc}\frac{1}{a+3}\le\frac{3}{8}\Leftrightarrow\frac{3(a+7-a)}{7(a+7)}-\sum_{cyc}\frac{1}{a+3}\le\frac{3}{8}$$

$$\Leftrightarrow\frac{3}{7}\sum_{cyc}\frac{a}{a+7}+\sum_{cyc}\frac{1}{a+3}\overset{(\bullet)}{\ge}\frac{51}{56}$$

Now, $$\sum_{cyc}\frac{a}{a+7}=\sum_{cyc}\frac{a^2}{a^2+7a}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab+7\sum_{cyc}a}$$

$$\overset{2\sum_{cyc}a+\sum_{cyc}ab=9}{=}\frac{\left(\sum_{cyc}a\right)^2}{\left(\sum_{cyc}a\right)^2-2\left(9-2\sum_{cyc}a\right)+7\sum_{cyc}a}$$

$$\therefore\frac{3}{7}\sum_{cyc}\frac{a}{a+7}\overset{(*)}{\ge}\frac{3t^2}{7(t^2+11t-18)}\left(t=\sum_{cyc}a\right)$$

$$9=2\sum_{cyc}a+\sum_{cyc}ab\overset{\text{Bergstrom}}{\ge}6\sqrt[3]{abc}+3\sqrt[3]{a^2b^2c^2}\Rightarrow x^2+2x-3\le 0$$

$$\left(x=\sqrt[3]{abc}\right)\Rightarrow(x-1)(x+3)\le 0\Rightarrow x=\sqrt[3]{abc}\le 1\Rightarrow abc\overset{(**)}{\le}1$$

Again, $$\sum_{cyc}\frac{1}{a+3}=\frac{\sum_{cyc}(b+3)(c+3)}{(a+3)(b+3)(c+3)}=\frac{\sum_{cyc}ab+6\sum_{cyc}a+27}{abc+27+9\sum_{cyc}a+3\sum_{cyc}ab}$$

$$\overset{\substack{2\sum_{cyc}a+\sum_{cyc}ab=9\\ \text{and}\\ \text{via }(**)}}{\ge}\frac{9-2\sum_{cyc}a+6\sum_{cyc}a+27}{1+27+9\sum_{cyc}a+3\left(9-2\sum_{cyc}a\right)}\therefore\sum_{cyc}\frac{1}{a+3}\overset{(***)}{\ge}\frac{36+4t}{55+3t}$$

$$\therefore(*)+(***)\Rightarrow\frac{3}{7}\sum_{cyc}\frac{a}{a+7}+\sum_{cyc}\frac{1}{a+3}\ge\frac{3t^2}{7(t^2+11t-18)}+\frac{36+4t}{55+3t}\overset{?}{\ge}\frac{51}{56}$$

$$\Leftrightarrow\frac{3t^2(55+3t)+7\left(t^2+11t-18\right)(36+4t)}{7(t^2+11t-18)(55+3t)}\overset{?}{\ge}\frac{51}{56}$$

$$\Leftrightarrow 143t^3+1312t^2-9957t+14202\overset{?}{\ge}0$$

$$\Leftrightarrow (t-3)\left(143t^2 + 1741(t-3) + 489\right) \overset{?}{\geq} 0 \to \text{true} \because 9 - 2\sum_{\text{cyc}} a = \sum_{\text{cyc}} ab$$

$$\leq \frac{1}{3}\left(\sum_{\text{cyc}} a\right)^2 \Rightarrow t^2 + 6t - 27 \geq 0 \Rightarrow (t+9)(t-3) \geq 0 \Rightarrow t \geq 3 \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \frac{a+1}{a^2+10a+21} + \frac{b+1}{b^2+10b+21} + \frac{c+1}{c^2+10c+21} \leq \frac{3}{16}$$

$$\forall\, a,b,c > 0 \,\middle|\, 2(a+b+c) + ab + bc + ca = 9, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$