ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $(x+y)(y+z)(z+x) = 64$, then prove that :

$$\frac{x}{(y+z)(x+2y)^2} + \frac{y}{(z+x)(y+2z)^2} + \frac{z}{(x+y)(z+2x)^2} \ge \frac{1}{24}$$

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$$\frac{x}{(y+z)(x+2y)^2} + \frac{y}{(z+x)(y+2z)^2} + \frac{z}{(x+y)(z+2x)^2}$$

$$= \frac{\left(\frac{x}{x+2y}\right)^2}{(xy+zx)} + \frac{\left(\frac{y}{y+2z}\right)^2}{(yz+xy)} + \frac{\left(\frac{z}{z+2x}\right)^2}{(zx+yz)} \overset{\text{Bergstrom}}{\ge} \frac{\left(\sum_{\text{cyc}} \frac{x}{x+2y}\right)^2}{2\sum_{\text{cyc}} xy} = \frac{\left(\sum_{\text{cyc}} \frac{x^2}{x^2+2xy}\right)^2}{2\sum_{\text{cyc}} xy}$$

$$\overset{\text{Bergstrom}}{\ge} \frac{\left(\frac{\left(\sum_{\text{cyc}} x\right)^2}{\sum_{\text{cyc}} x^2 + 2\sum_{\text{cyc}} xy}\right)^2}{2\sum_{\text{cyc}} xy} = \frac{\left(\frac{\left(\sum_{\text{cyc}} x\right)^2}{\left(\sum_{\text{cyc}} x\right)^2}\right)^2}{2\sum_{\text{cyc}} xy} \overset{?}{\ge} \frac{1}{24} \Leftrightarrow \sum_{\text{cyc}} xy \overset{?}{\le} 12$$

$$\overset{(x+y)(y+z)(z+x)=64}{=} \frac{12}{16} \cdot \sqrt[3]{(x+y)^2(y+z)^2(z+x)^2}$$

$$\Leftrightarrow 27 \prod_{\text{cyc}} (x+y)^2 \underset{(*)}{\overset{?}{\ge}} 64 \left(\sum_{\text{cyc}} xy\right)^3$$

$$\text{Now, } \left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} xy\right) \overset{A-G}{\ge} \left(3.\sqrt[3]{xyz}\right)\left(3.\sqrt[3]{x^2y^2z^2}\right) = 9xyz$$

$$\Rightarrow 9\left(\left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} xy\right) - xyz\right) \ge 8\left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} xy\right)$$

$$\Rightarrow 9 \prod_{\text{cyc}} (x+y) \ge 8\left(\sum_{\text{cyc}} x\right)\left(\sum_{\text{cyc}} xy\right)$$

$$\Rightarrow 81 \prod_{\text{cyc}} (x+y)^2 \ge 64\left(\sum_{\text{cyc}} x\right)^2\left(\sum_{\text{cyc}} xy\right)^2 \ge 64\left(3\sum_{\text{cyc}} xy\right)\left(\sum_{\text{cyc}} xy\right)^2$$

$$\Rightarrow 27 \prod_{\text{cyc}} (x+y)^2 \ge 64\left(\sum_{\text{cyc}} xy\right)^3 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{x}{(y+z)(x+2y)^2} + \frac{y}{(z+x)(y+2z)^2} + \frac{z}{(x+y)(z+2x)^2} \ge \frac{1}{24}$$

$$\forall\, x, y, z > 0 \mid (x+y)(y+z)(z+x) = 64, \text{''} = \text{''} \text{ iff } x = y = z = 2 \text{ (QED)}$$