ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 1$ and $a + b + c = 6$, then prove that :

$$\frac{a}{a^2 - a + 1} + \frac{b}{b^2 - b + 1} + \frac{c}{c^2 - c + 1} \geq 2$$

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Let $a - 1 = x, b - 1 = y, c - 1 = z$ and then : $\sum_{cyc} \frac{a}{a^2 - a + 1}$

$$= \sum_{cyc} \frac{a}{(a-1)^2 + a} = \sum_{cyc} \frac{x+1}{x^2 + x + 1} = \sum_{cyc} \frac{x + 1 + x^2 - x^2}{x^2 + x + 1}$$

$$= 3 - \sum_{cyc} \frac{x^2}{x^2 + x + 1} \overset{A-G}{\geq} 3 - \sum_{cyc} \frac{x^2}{2x + x} = 3 - \frac{1}{3}\sum_{cyc} x = 3 - \frac{1}{3}.3$$

$$\left(\because \sum_{cyc} x = \sum_{cyc} (a-1) \overset{a+b+c=6}{=} 6 - 3 = 3 \right) = 2$$

$$\therefore \frac{a}{a^2 - a + 1} + \frac{b}{b^2 - b + 1} + \frac{c}{c^2 - c + 1} \geq 2 \ \forall\, a, b, c > 1 \,\Big|\, \sum_{cyc} a = 3,$$

$'' ='' $ iff $a = b = c = 2$ (QED)