ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > \frac{1}{3}$ and $a + b + c = 3$, then prove that :

$$\frac{1}{3a^2 - 3a + 1} + \frac{1}{3b^2 - 3b + 1} + \frac{1}{3c^2 - 3c + 1} \geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{3a^2 - 3a + 1} = \sum_{cyc} \frac{3}{9a^2 - 9a + 3} = \sum_{cyc} \frac{3}{(3a-1)^2 - (3a-1) + 1}$$

$$= \sum_{cyc} \frac{3}{x^2 - x + 1} \ (x = 3a - 1, y = 3b - 1, z = 3c - 1 \ and \ x, y, z > 0) \geq 3$$

$$\Leftrightarrow \sum_{cyc} (y^2 - y + 1)(z^2 - z + 1) \geq \prod_{cyc} (x^2 - x + 1)$$

$$\Leftrightarrow xyz \sum_{cyc} xy + xyz + \sum_{cyc} x^2 + 2 - x^2y^2z^2 - xyz \sum_{cyc} x - \sum_{cyc} x \overset{(*)}{\geq} 0$$

$$\text{Now}, \because \sum_{cyc} x = \sum_{cyc} (3a - 1) \overset{a+b+c=3}{=} 6 \therefore \text{LHS of } (*) \overset{A-G}{\geq}$$

$$3(xyz)^{\frac{5}{3}} + xyz + \frac{1}{3}.36 + 2 - x^2y^2z^2 - 6xyz \sum_{cyc} x - 6 \overset{?}{\geq} 0$$

$$\Leftrightarrow 3t^5 - 5t^3 + 8 - t^6 \overset{?}{\geq} 0 \ (t = \sqrt[3]{xyz}) \Leftrightarrow t^6 - 3t^5 + 5t^3 - 8 \overset{?}{\leq} 0$$

$$\Leftrightarrow (t - 2)(t^5 - t^4 - 2t^3 + t^2 + 2t + 4) \underbrace{\overset{?}{\leq}}_{(**)} 0$$

$$\text{Now}, t^4 - t^3 - 2t^2 + t + 2 = (t^4 - 2t^2 + 1) - t(t^2 - 1) + 1$$

$$= (t^2 - 1)^2 - (t - 1 + 1)(t^2 - 1) + 1 = (t^2 - 1)^2 - (t - 1)^2(t + 1) - t^2 + 2$$

$$= (t - 1)^2(t^2 + t) - t^2 + 2 = t(t - 1)^2 + t^2(t^2 - 2t + 1) - t^2 + 2$$

$$= t(t - 1)^2 + \frac{1}{16}(16t^4 - 32t^3 + 32)$$

$$= t(t - 1)^2 + \frac{1}{16}\left((4t^2 + 4t + 3)(2t - 3)^2 + 5\right) > 0$$

$$\Rightarrow t^4 - t^3 - 2t^2 + t + 2 > 0 \rightarrow (1)$$

$$\text{Again}, t = \sqrt[3]{xyz} \overset{A-G}{\leq} \frac{1}{3} \sum_{cyc} x = \frac{1}{3}.6 \Rightarrow t - 2 \leq 0 \rightarrow (2) \therefore (1), (2) \Rightarrow \text{LHS of } (**) =$$

$$(t - 2)\left(t(t^4 - t^3 - 2t^2 + t + 2) + 4\right) \leq 0 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{3a^2 - 3a + 1} + \frac{1}{3b^2 - 3b + 1} + \frac{1}{3c^2 - 3c + 1} \geq 3 \ \forall \ a, b, c > \frac{1}{3} \mid \sum_{cyc} a = 3,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$