ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 2$ and $a + b + c = 9$, then prove that :

$$\frac{1}{a^2 - 4a + 5} + \frac{1}{b^2 - 4b + 5} + \frac{1}{c^2 - 4c + 5} \geq \frac{3}{2}$$

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$$\text{Let } a - 2 = x, b - 2 = y, c - 2 = z \text{ and then : } \sum_{\text{cyc}} \frac{1}{a^2 - 4a + 5}$$

$$= \sum_{\text{cyc}} \frac{1}{(a-2)^2 + 1} = \sum_{\text{cyc}} \frac{1}{x^2 + 1} = \sum_{\text{cyc}} \frac{1 + x^2 - x^2}{x^2 + 1} = 3 - \sum_{\text{cyc}} \frac{x^2}{x^2 + 1} \overset{\text{A-G}}{\geq}$$

$$3 - \sum_{\text{cyc}} \frac{x^2}{2x} = 3 - \frac{1}{2} \sum_{\text{cyc}} x = 3 - \frac{1}{2}.3 \left(\because \sum_{\text{cyc}} x = \sum_{\text{cyc}} (a-2) \overset{a+b+c=9}{=} 9 - 6 = 3 \right)$$

$$= \frac{3}{2} \therefore \frac{1}{a^2 - 4a + 5} + \frac{1}{b^2 - 4b + 5} + \frac{1}{c^2 - 4c + 5} \geq \frac{3}{2} \; \forall \, a, b, c > 2 \,\Big|\, \sum_{\text{cyc}} a = 9,$$

$$'' ='' \text{ iff } a = b = c = 3 \text{ (QED)}$$