ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > \frac{4}{3}$ and $a + \mathrm{b} + \mathrm{c} = 6$, then prove that :

$$\frac{a}{a^2+1} + \frac{\mathrm{b}}{\mathrm{b}^2+1} + \frac{\mathrm{c}}{\mathrm{c}^2+1} \geq \frac{6}{5}$$

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$$\sum_{\text{cyc}} \frac{a}{a^2+1} = \sum_{\text{cyc}} \frac{9a}{9a^2+9} = \sum_{\text{cyc}} \frac{3(3a-4+4)}{9a^2-24a+16+24a-7}$$

$$= \sum_{\text{cyc}} \frac{3(3a-4+4)}{(3a-4)^2+8(3a-4)+25}$$

$$= \sum_{\text{cyc}} \frac{3(x+4)}{x^2+8x+25}\ (x = 3a-4 > 0 \text{ and analogs}) \geq \frac{6}{5}$$

$$\Leftrightarrow \sum_{\text{cyc}} \frac{x+4}{x^2+8x+25} \overset{(*)}{\geq} \frac{2}{5}$$

$$\text{Now, } \frac{x+4}{x^2+8x+25} \overset{?}{\geq} \frac{12-x}{75} \Leftrightarrow 75(x+4) \overset{?}{\geq} (12-x)\left(x^2+8x+25\right)$$

$$\left(\sum_{\text{cyc}} x = \sum_{\text{cyc}} (3a-4) \overset{a+\mathrm{b}+\mathrm{c}=6}{=} 6 \Rightarrow x < 6 < 12 \Rightarrow 12 - x > 0\right)$$

$$\Leftrightarrow x(x-2)^2 \overset{?}{\geq} 0 \to \text{true} \therefore \frac{x+4}{x^2+8x+25} \geq \frac{12-x}{75} \text{ and analogs}$$

$$\Rightarrow \sum_{\text{cyc}} \frac{x+4}{x^2+8x+25} \geq \sum_{\text{cyc}} \frac{12-x}{75} = \frac{36-\sum_{\text{cyc}} x}{75} = \frac{36-6}{75} = \frac{2}{5} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a}{a^2+1} + \frac{\mathrm{b}}{\mathrm{b}^2+1} + \frac{\mathrm{c}}{\mathrm{c}^2+1} \geq \frac{6}{5}\ \forall\, a, \mathrm{b}, \mathrm{c} > \frac{4}{3} \,\Big|\, \sum_{\text{cyc}} a = 6,$$

$$'' ='' \text{ iff } a = \mathrm{b} = \mathrm{c} = 1 \text{ (QED)}$$