# ROMANIAN MATHEMATICAL MAGAZINE

**If $x, y > 0$ and $x + y + xy = 3$, then prove that :**

$$\sqrt{9-x^2}+\sqrt{9-y^2}+\frac{x+y}{4}\le\frac{1+8\sqrt{2}}{2}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumva Chakraborty-Kolkata-India*

$$xy = 3-(x+y)\le\frac{(x+y)^2}{4}\Rightarrow t^2\ge 12-4t\ (t=x+y)$$

$$\Rightarrow t^2+4t-12\ge 0\Rightarrow (t+6)(t-2)\ge 0\Rightarrow t\ge 2\to (1)$$

$$\text{Now, }\sqrt{9-x^2}+\sqrt{9-y^2}\overset{\text{CBS}}{\le}\sqrt{2}.\sqrt{18-((x+y)^2-2xy)}\overset{x+y+xy=3}{=}$$

$$\sqrt{2}.\sqrt{18-\left((x+y)^2-2\left(3-(x+y)\right)\right)}=\sqrt{2}.\sqrt{18-(t^2-6+2t)}$$

$$=\sqrt{2}.\sqrt{(4-t)(t+6)}=\sqrt{2}.\sqrt{2(4-t).\left(\frac{t+6}{2}\right)}\overset{A-G}{\le}\sqrt{2}.\left(\frac{8-2t+\frac{t+6}{2}}{2}\right)$$

$$\left(\text{note}: 2\overset{\text{via }(1)}{\le}t=3-xy<3<4\right)=\sqrt{2}.\frac{22-3t}{4}\Rightarrow\sqrt{9-x^2}+\sqrt{9-y^2}+\frac{x+y}{4}$$

$$\le\sqrt{2}.\frac{22-3t}{4}+\frac{t}{4}=\frac{11\sqrt{2}}{2}-\frac{(3\sqrt{2}-1)t}{4}\overset{\text{via }(1)}{\le}\frac{11\sqrt{2}}{2}-\frac{3\sqrt{2}-1}{2}$$

$$\therefore\sqrt{9-x^2}+\sqrt{9-y^2}+\frac{x+y}{4}\le\frac{1+8\sqrt{2}}{2}\ \forall\, x,y>0\,\middle|\,x+y+xy=3,$$

$$''='' \text{ iff } x=y=1\ (\text{QED})$$