ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a^2 + \mathrm{b}^2 + \mathrm{c}^2 = 3$, then prove that :

$$\frac{a^3}{\mathrm{b}^2 + \mathrm{c}^2} + \frac{\mathrm{b}^3}{\mathrm{c}^2 + a^2} + \frac{\mathrm{c}^3}{a^2 + \mathrm{b}^2} \ge \frac{3}{2}$$

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$$\sum_{\text{cyc}} \frac{a^3}{\mathrm{b}^2 + \mathrm{c}^2} \overset{\text{Chebyshev}}{\ge} \frac{1}{3}\left(\sum_{\text{cyc}} \frac{a^2}{\mathrm{b}^2 + \mathrm{c}^2}\right)\left(\sum_{\text{cyc}} a\right)$$

$$\left(\text{WLOG assuming } a \ge \mathrm{b} \ge \mathrm{c} \Rightarrow \frac{a^2}{\mathrm{b}^2 + \mathrm{c}^2} \ge \frac{\mathrm{b}^2}{\mathrm{c}^2 + a^2} \ge \frac{\mathrm{c}^2}{a^2 + \mathrm{b}^2}\right)$$

$$= \frac{1}{3}\left(\sum_{\text{cyc}} \frac{a^4}{a^2\mathrm{b}^2 + \mathrm{c}^2 a^2}\right)\left(\sum_{\text{cyc}} a\right) \overset{\text{Bergstrom}}{\ge} \frac{1}{3} \cdot \frac{\left(\sum_{\text{cyc}} a^2\right)^2}{2\sum_{\text{cyc}} a^2\mathrm{b}^2} \cdot \sum_{\text{cyc}} a \overset{a^2 + \mathrm{b}^2 + \mathrm{c}^2 = 3}{=}$$

$$\frac{1}{\sqrt{3}} \cdot \frac{\left(\sum_{\text{cyc}} a^2\right)^2}{2\sum_{\text{cyc}} a^2\mathrm{b}^2} \cdot \frac{\sum_{\text{cyc}} a}{\sqrt{\sum_{\text{cyc}} a^2}} \overset{?}{\ge} \frac{3}{2} \Leftrightarrow \left(\sum_{\text{cyc}} a\right)^2 \left(\sum_{\text{cyc}} a^2\right)^3 \boxed{\overset{?}{\underset{(*)}{\ge}}} 27\left(\sum_{\text{cyc}} a^2\mathrm{b}^2\right)^2$$

Assigning $\mathrm{b} + \mathrm{c} = x, \mathrm{c} + a = \mathrm{y}, a + \mathrm{b} = \mathrm{z} \Rightarrow x + \mathrm{y} - \mathrm{z} = 2\mathrm{c} > 0, y + z - x = 2a > 0$ and $\mathrm{z} + x - \mathrm{y} = 2\mathrm{b} > 0 \Rightarrow x + \mathrm{y} > z, \mathrm{y} + z > x, \mathrm{z} + x > y \Rightarrow x, \mathrm{y}, \mathrm{z}$ form sides of a triangle with semiperimeter, circumradius and inradius $= \mathrm{s}, \mathrm{R}, \mathrm{r}$ (say) yielding $2\sum_{\text{cyc}} a = \sum_{\text{cyc}} x = 2\mathrm{s} \Rightarrow \sum_{\text{cyc}} a = \mathrm{s} \rightarrow (1) \Rightarrow a = \mathrm{s} - x, \mathrm{b} = \mathrm{s} - \mathrm{y}, \mathrm{c} = \mathrm{s} - \mathrm{z}$

$$\therefore a\mathrm{bc} = \mathrm{r}^2\mathrm{s} \rightarrow (2) \text{ and such substitutions} \Rightarrow \sum_{\text{cyc}} a\mathrm{b} = \sum_{\text{cyc}} (\mathrm{s} - x)(\mathrm{s} - \mathrm{y})$$

$$\Rightarrow \sum_{\text{cyc}} a\mathrm{b} = 4\mathrm{Rr} + \mathrm{r}^2 \rightarrow (3), \sum_{\text{cyc}} a^2 = \left(\sum_{\text{cyc}} a\right)^2 - 2\sum_{\text{cyc}} a\mathrm{b} \overset{\text{via (1) and (3)}}{=}$$

$$\mathrm{s}^2 - 2(4\mathrm{Rr} + \mathrm{r}^2) \Rightarrow \sum_{\text{cyc}} a^2 = \mathrm{s}^2 - 8\mathrm{Rr} - 2\mathrm{r}^2 \rightarrow (4),$$

$$\sum_{\text{cyc}} a^2\mathrm{b}^2 = \left(\sum_{\text{cyc}} a\mathrm{b}\right)^2 - 2a\mathrm{bc}\left(\sum_{\text{cyc}} a\right) \overset{\text{via (1),(2) and (3)}}{=} \left(4\mathrm{Rr} + \mathrm{r}^2\right)^2 - 2\mathrm{r}^2\mathrm{s}.\mathrm{s}$$

$$\Rightarrow \sum_{\text{cyc}} a^2\mathrm{b}^2 = \mathrm{r}^2\left((4\mathrm{R} + \mathrm{r})^2 - 2\mathrm{s}^2\right) \rightarrow (5)$$

$$\text{Via } (1), (4), (5), (*) \Leftrightarrow \mathrm{s}^2\left(\mathrm{s}^2 - 8\mathrm{Rr} - 2\mathrm{r}^2\right)^3 \ge 27\mathrm{r}^4\left((4\mathrm{R} + \mathrm{r})^2 - 2\mathrm{s}^2\right)^2$$

$$\Leftrightarrow \mathrm{s}^8 - (24\mathrm{Rr} - 6\mathrm{r}^2)\mathrm{s}^6 + \mathrm{r}^2(192\mathrm{R}^2 + 96\mathrm{Rr} - 96\mathrm{r}^2)\mathrm{s}^4$$

$$-\mathrm{r}^3(512\mathrm{R}^3 - 1344\mathrm{R}^2\mathrm{r} - 768\mathrm{Rr}^2 - 100\mathrm{r}^3)\mathrm{s}^2 - 27\mathrm{r}^4(4\mathrm{R} + \mathrm{r})^4 \boxed{\overset{(**)}{\ge}} 0 \text{ and}$$

$$\because \left(\mathrm{s}^2 - 16\mathrm{Rr} + 5\mathrm{r}^2\right)^4 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (**), \text{ it suffices to prove :}$$

$$\text{LHS of } (**) \geq \left(s^2 - 16Rr + 5r^2\right)^4$$

$$\Leftrightarrow (20R - 13r)s^6 - r\left(672R^2 - 528Rr + 123r^2\right)s^4$$
$$+r^2\left(7936R^3 - 7008R^2r + 2784Rr^2 - 200r^3\right)s^2$$
$$-r^3\left(36224R^4 - 37504R^3r + 20496R^2r^2 - 3784Rr^3 + 326r^4\right) \boxed{\overset{(***)}{\geq}} 0$$

$$and \because (20R - 13r)\left(s^2 - 16Rr + 5r^2\right)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***),$$
$$\text{it suffices to prove : LHS of } (***) \geq (20R - 13r)\left(s^2 - 16Rr + 5r^2\right)^3$$
$$\Leftrightarrow \left(288R^2 - 396Rr + 72r^2\right)s^4 - r\left(7424R^3 - 12576R^2r + 4956Rr^2 - 775r^3\right)s^2$$
$$+r^2\left(45696R^4 - 92544R^3r + 53424R^2r^2 - 14316Rr^3 + 1299r^4\right) \boxed{\overset{(****)}{\geq}} 0$$

$$and \because \left(288R^2 - 396Rr + 72r^2\right)\left(s^2 - 16Rr + 5r^2\right)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$
$$\text{to prove } (***), \text{it suffices to prove : LHS of } (****) \geq$$
$$\left(288R^2 - 396Rr + 72r^2\right)\left(s^2 - 16Rr + 5r^2\right)^2$$
$$\Leftrightarrow \left(1792R^3 - 2976R^2r + 1308Rr^2 + 55r^3\right)s^2 \boxed{\overset{(*****)}{\geq}}$$
$$r\left(28032R^4 - 54912R^3r + 35568R^2r^2 - 7104Rr^3 + 501r^4\right)$$

$$\text{Now, LHS of } (*****) \overset{\text{Gerretsen}}{\geq} \left(1792R^3 - 2976R^2r + 1308Rr^2 + 55r^3\right)\binom{16Rr}{-5r^2}$$
$$\overset{?}{\geq} r\left(28032R^4 - 54912R^3r + 35568R^2r^2 - 7104Rr^3 + 501r^4\right)$$
$$\Leftrightarrow 160t^4 - 416t^3 + 60t^2 + 361t - 194 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\Big((t-2)\left(160t^2 + 224t + 316\right) + 729\Big) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\Rightarrow (*****) \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \frac{a^3}{b^2 + c^2} + \frac{b^3}{c^2 + a^2} + \frac{c^3}{a^2 + b^2} \geq \frac{3}{2} \;\forall\, a, b, c > 0 \,\big|\, a^2 + b^2 + c^2 = 3,$$
$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$