ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a + b + c = 3$, then prove that :

$$\frac{a^3}{9-a^2}+\frac{b^3}{9-b^2}+\frac{c^3}{9-c^2}\geq\frac{3}{8}$$

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$$\sum_{cyc}\frac{a^3}{9-a^2}=\sum_{cyc}\frac{a^3}{(3-a)(3+a)}\overset{a+b+c=3}{=}\sum_{cyc}\frac{a^3}{(b+c)\big((a+b)+(c+a)\big)}$$

$$\overset{\text{Holder}}{\geq}\frac{\left(\sum_{cyc}a\right)^3}{3\sum_{cyc}\big((b+c)(a+b)\big)+3\sum_{cyc}\big((b+c)(c+a)\big)}$$

$$=\frac{\left(\sum_{cyc}a\right)^3}{3\left(\sum_{cyc}a^2+3\sum_{cyc}ab\right)+3\left(\sum_{cyc}a^2+3\sum_{cyc}ab\right)}=\frac{\left(\sum_{cyc}a\right)^3}{6\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)}$$

$$\geq\frac{\left(\sum_{cyc}a\right)^3}{6\left(\left(\sum_{cyc}a\right)^2+\frac{\left(\sum_{cyc}a\right)^2}{3}\right)}=\frac{3}{8}\therefore\frac{a^3}{9-a^2}+\frac{b^3}{9-b^2}+\frac{c^3}{9-c^2}\geq\frac{3}{8}$$

$$\forall\, a,b,c>0\,\big|\,a+b+c=3,''='' \text{ iff } a=b=c=1\ (\text{QED})$$