ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z \in \mathbb{R}$ and $x^2 + y^2 + z^2 = 3$, then prove that :
$8(2 - x)(2 - y)(2 - z) \geq (x + yz)(y + zx)(z + xy)$

Proposed by Nguyen Hung Cuong-Vietnam
Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case : exactly two among } x, y, z \text{ equal zero}}$ and then : LHS >

$$8\left(2-\sqrt{3}\right)^3 \left(\begin{array}{c} \because x^2, y^2, z^2 < 3 \\ \therefore -\sqrt{3} < x, y, z < \sqrt{3} \end{array} \Rightarrow (2-x), (2-y), (2-z) > 2-\sqrt{3}\right) > 0$$
$$= \text{RHS}$$

$\boxed{\text{Case : exactly one among } x, y, z \text{ equals zero}}$ WLOG we may assume $x = 0$ and so, $y, z \neq 0$ with $y^2 + z^2 = 3$ and we are to prove : $16(2-y)(2-z) \geq y^2z^2 \to (1)$

$$\boxed{\text{Sub - case : } yz < 0} \therefore \left\{\begin{array}{c} y > 0, z < 0 \\ \text{or} \\ y < 0, z > 0 \end{array}\right\} \text{ and } \left\{\begin{array}{c} \because y^2, z^2 < 3 \\ \therefore -\sqrt{3} < y, z < \sqrt{3} \end{array}\right\} \therefore \text{LHS of (1)} >$$
$$32\left(2-\sqrt{3}\right) \left(\because \min\{(2-y), (2-z)\} > 2-\sqrt{3} \text{ and } \max\{(2-y), (2-z)\} > 2\right)$$
$$\approx 8.574374 > \frac{9}{4} \geq y^2z^2 \left(\because 3 = |y|^2 + |z|^2 \overset{A-G}{\geq} 2|yz| \Rightarrow |yz| \leq \frac{3}{2}\right)$$
$$\Rightarrow (1) \text{ is true (strict inequality)}$$

$$\boxed{\text{Sub - case : } yz > 0} \text{ and } (1) \Leftrightarrow 64 - 32(y+z) + 16yz \geq y^2z^2$$
$$\Leftrightarrow 64 + yz(16 - yz) \geq 32(y+z) \to (a)$$

$$\text{Now, } 0 < yz = |yz| \leq \frac{3}{2} < 16 \therefore yz(16 - yz) > 0 \Rightarrow \text{LHS of (i)} > 64 \text{ and if}$$
$y + z < 0$, then (a) is definitely true and so, we focus on : $y + z > 0$ and then :
$$(a) \overset{y^2+z^2=3}{\Leftrightarrow} \left(64 + t(16-t)\right)^2 \geq 1024(3 + 2t) \; (t = yz) \Leftrightarrow$$
$$t^4 - 32t^3 + 128t^2 + 1024 \geq 0 \Leftrightarrow (t-6)\left(t^3 - 26t^2 - 28t - 168\right) + 16 \geq 0 \text{ and}$$
$$\because t \leq \frac{3}{2} < 6 \therefore \text{it suffices to prove : } t^3 - 26t^2 - 28t - 168 < 0$$
$$\Leftrightarrow 4t^3 - 104t^2 - 112t - 672 < 0 \Leftrightarrow (t-23)(2t-3)^2 - 397t - 465 < 0 \to \textit{true}$$
$$\because 0 < t \leq \frac{3}{2} < 23 \Rightarrow (a) \Rightarrow (1) \text{ is true (strict inequality)}$$

We now focus on the cases when none of x, y, z equals zero

$$\boxed{\text{Case : } x, y, z < 0} \text{ and then : } 8(2-x)(2-y)(2-z) > 8\left(2^3\right) = 64 \to (2)$$
$$\text{and } (x+yz)(y+zx)(z+xy) = xyz + x^2y^2z^2 + xyz\sum_{\text{cyc}} x^2 + \sum_{\text{cyc}} x^2y^2$$
$$\overset{x^2+y^2+z^2=3}{=} 4xyz + x^2y^2z^2 + \sum_{\text{cyc}} x^2y^2 \leq 4xyz + x^2y^2z^2 + \frac{1}{3}\left(\sum_{\text{cyc}} x^2\right)^2$$
$$\underset{\because xyz<0}{\overset{x^2+y^2+z^2=3 \text{ and}}{<}} x^2y^2z^2 + 3 \leq 1 + 3$$
$$\left(\because 3 = |x|^2 + |y|^2 + |z|^2 \overset{A-G}{\geq} 3\sqrt[3]{|xyz|^2} \Rightarrow |xyz|^2 = x^2y^2z^2 \leq 1\right) = 4 < 64$$
$$\overset{\text{via (2)}}{<} 8(2-x)(2-y)(2-z)$$

$$\therefore 8(2-x)(2-y)(2-z) > (x+yz)(y+zx)(z+xy)$$

$\boxed{\text{Case : exactly two among } x,y,z<0}$ and WLOG we may assume $y,z<0$ and $\because$

$$0<x<\sqrt{3} \therefore 8(2-x)(2-y)(2-z) > 8(2-\sqrt{3})(2^2) = 32(2-\sqrt{3}) \to (3)$$

$$\text{and } (x+yz)(y+zx)(z+xy) = 4xyz + x^2y^2z^2 + \sum_{cyc} x^2y^2 \leq$$

$$4xyz + x^2y^2z^2 + \frac{1}{3}\left(\sum_{cyc} x^2\right)^2 \overset{x^2+y^2+z^2=3}{\leq} 4xyz + x^2y^2z^2 + 3 \overset{x^2y^2z^2\leq 1}{\leq}$$

$$4+1+3 \left(\begin{matrix}\because xyz>0 \therefore x^2y^2z^2 \leq 1 \\ \Rightarrow xyz \leq 1\end{matrix}\right) \Rightarrow (x+yz)(y+zx)(z+xy) < 8 \to (4)$$

$$\therefore (3),(4) \Rightarrow \text{it suffices to prove : } 32(2-\sqrt{3}) > 8 \Leftrightarrow 7 > 4\sqrt{3} \Leftrightarrow 49 > 48$$

$$\therefore 8(2-x)(2-y)(2-z) > (x+yz)(y+zx)(z+xy)$$

$\boxed{\text{Case : exactly one among } x,y,z<0}$ and $\text{LHS} = 8\left(8 - xyz + 2\sum_{cyc} xy - 4\sum_{cyc} x\right)$

$$\overset{x^2+y^2+z^2=3}{=} 8\left(5 + \left(\sum_{cyc} x^2 + 2\sum_{cyc} xy\right) - xyz - 4\sum_{cyc} x\right)$$

$$= 8\left(\left(\sum_{cyc} x\right)^2 - 4\sum_{cyc} x + 4 + 1 - xyz\right) = 8\left(\sum_{cyc} x - 2\right)^2 + 8 - 8xyz \overset{xyz<0}{>} 8$$

$$\therefore \text{LHS} > 8 \to (5) \text{ and } (x+yz)(y+zx)(z+xy) \leq 4xyz + x^2y^2z^2 + 3 \overset{\substack{x^2y^2z^2\leq 1 \\ \text{and} \\ xyz<0}}{\leq} 4$$

$$< 8 \overset{\text{via }(5)}{<} \text{LHS} \therefore 8(2-x)(2-y)(2-z) > (x+yz)(y+zx)(z+xy)$$

$\boxed{\text{Case : } x,y,z>0}$ and then : $8(2-x)(2-y)(2-z) \geq (x+yz)(y+zx)(z+xy)$

$$\Leftrightarrow 64 + 16\sum_{cyc} xy - 32\sum_{cyc} x \geq 12xyz + x^2y^2z^2 + \sum_{cyc} x^2y^2$$

$$\overset{x^2+y^2+z^2=3}{\Leftrightarrow} \frac{64}{3}\sum_{cyc} x^2 + +16\sum_{cyc} xy \geq$$

$$\frac{12\sqrt{3}xyz}{\sqrt{\sum_{cyc} x^2}} + \frac{32(\sum_{cyc} x).\sqrt{\sum_{cyc} x^2}}{\sqrt{3}} + \frac{9x^2y^2z^2}{\left(\sum_{cyc} x^2\right)^2} + \frac{3\sum_{cyc} x^2y^2}{\sum_{cyc} x^2}$$

$$\Leftrightarrow \frac{64\sum_{cyc} x^2 + 48\sum_{cyc} xy}{3} - \frac{9x^2y^2z^2 + 3(\sum_{cyc} x^2)(\sum_{cyc} x^2y^2)}{\left(\sum_{cyc} x^2\right)^2}$$

$$\geq \frac{36xyz + 32(\sum_{cyc} x)(\sum_{cyc} x^2)}{\sqrt{3\sum_{cyc} x^2}}$$

$$\Leftrightarrow \frac{\left((64\sum_{cyc} x^2 + 48\sum_{cyc} xy)(\sum_{cyc} x^2)^2 - 27x^2y^2z^2 - 9(\sum_{cyc} x^2)(\sum_{cyc} x^2y^2)\right)^2}{9\left(\sum_{cyc} x^2\right)^4}$$

$$\overset{(*)}{\geq} \frac{\left(36xyz + 32(\sum_{cyc} x)(\sum_{cyc} x^2)\right)^2}{3\sum_{cyc} x^2}$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ *and* $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ *form* sides of *a* triangle with semiperimeter, circumradius *and* inradius

$$= s, R, r \ (say) \text{ yielding } 2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \to (\bullet)$$

$$\Rightarrow x = s - X, y = s - Y, z = s - Z \therefore xyz = r^2 s \to (\bullet\bullet) \text{ and such}$$

$$\text{substitutions} \Rightarrow \sum_{cyc} xy = \sum_{cyc} (s - X)(s - Y) \Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (\bullet\bullet\bullet)$$

$$\text{and } \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \to (\bullet\bullet\bullet\bullet) \text{ and also, } \sum_{cyc} x^2y^2$$

$$= \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via } (\bullet),(\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{=} (4Rr + r^2)^2 - 2r^2 s.s \Rightarrow \sum_{cyc} x^2y^2$$

$$= r^2((4R + r)^2 - 2s^2) \to (\bullet\bullet\bullet\bullet\bullet) \therefore (\bullet), (\bullet\bullet), (\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet\bullet) \Rightarrow (*) \Leftrightarrow$$

$$\frac{\begin{pmatrix} \left(64(s^2 - 8Rr - 2r^2) + 48(4Rr + r^2)\right)(s^2 - 8Rr - 2r^2)^2 - 27r^4s^2 \\ -9(s^2 - 8Rr - 2r^2)\left(r^2((4R + r)^2 - 2s^2)\right) \end{pmatrix}^2}{9(s^2 - 8Rr - 2r^2)^4}$$

$$\geq \frac{\left(36r^2 s + 32s(s^2 - 8Rr - 2r^2)\right)^2}{3(s^2 - 8Rr - 2r^2)}$$

$$\Leftrightarrow 256s^{12} - (12288Rr + 4224r^2)s^{10} + r^2(250368R^2 + 163776Rr + 23541r^2)s^8$$
$$-r^3(2782720R^3 + 2572176R^2r + 724152Rr^2 + 64000r^3)s^6$$
$$r^4(17835072R^4 + 20546880R^3r + 8418096R^2r^2 + 1473144Rr^3 + 93714r^4)s^4$$
$$-r^5\begin{pmatrix} 62505984R^5 + 83818752R^4r + 43993920R^3r^2 \\ +11329968R^2r^3 + 1433976Rr^4 + 71400r^5 \end{pmatrix}s^2$$
$$+r^6\begin{pmatrix} 93392896R^6 + 140089344R^5r + 87555840R^4r^2 \\ +29185280R^3r^3 + 5472240R^2r^4 + 547224Rr^5 + 22801r^6 \end{pmatrix} \overset{(**)}{\geq} 0$$

$$\text{Now, } 688768t^3 - 1977828t^2 + 1834338t - 557705 \ \left(t = \frac{R}{r}\right)$$

$$= (t - 2)(688768t^2 - 600292t + 633754) + 709803 \overset{\text{Euler}}{\geq} 709803 > 0,$$

$$1073220t^4 - 4106172t^3 + 5683266t^2 - 3426096t + 766554$$

$$= (t - 2)\begin{pmatrix} 1073220t^3 - 1959732t^2 \\ +1763802t + 101508 \end{pmatrix} + 969570 \overset{\text{Euler}}{\geq} 969570 > 0 \text{ and}$$

$$3574848t^5 - 17142456t^4 + 31580580t^3 - 28406220t^2 + 12606927t$$

$$-2223690 = (t - 2)\begin{pmatrix} (t - 2)\begin{pmatrix} 3574848t^3 - 2843064t^2 \\ +5908932t + 6601764 \end{pmatrix} \\ +15378255 \end{pmatrix} + 2125764$$

$$\overset{\text{Euler}}{\geq} 2125764 > 0 \therefore P = 256(s^2 - 16Rr + 5r^2)^6$$

$$+(12288Rr - 11904r^2)(s^2 - 16Rr + 5r^2)^5$$

$$+r^2(250368R^2-481344Rr+225141r^2)(s^2-16Rr+5r^2)^4$$
$$+4r^3(688768R^3-1977828R^2r+1834338Rr^2-557705r^3)(s^2-16Rr+5r^2)^3$$
$$+16r^4\begin{pmatrix}1073220R^4-4106172R^3r+5683266R^2r^2\\-3426096Rr^3+766554r^4\end{pmatrix}(s^2-16Rr+5r^2)^2+$$
$$16r^5\begin{pmatrix}3574848R^5-17142456R^4r+31580580R^3r^2\\-28406220R^2r^3+12606927Rr^4-2223690r^5\end{pmatrix}(s^2-16Rr+5r^2)$$
$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$
$$\boxed{\text{LHS of } (**) \geq P}$$
$$\Leftrightarrow 2476160t^6-14330976t^5+33045330t^4-39546092t^3$$
$$+26192019t^2-9169998t+1332368\geq 0$$
$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}262992t^4+2213168t^3(t-2)+5435346t^2\\-99364t+4053179\end{pmatrix}+7440174\right)$$
$$\geq 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore 8(2-x)(2-y)(2-z)\geq(x+yz)(y+zx)(z+xy)\ \forall\, x,y,z>0$$
$$\textit{and } \text{so, combining } \textit{all} \text{ cases, } 8(2-x)(2-y)(2-z)\geq(x+yz)(y+zx)(z+xy)$$
$$\forall\, x,y,z\in\mathbb{R}\,\middle|\, x^2+y^2+z^2=3, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$