ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } a,b>0 \text{ and } a+b=ab, \text{then prove that}:$$
$$\frac{1}{a^2+2a}+\frac{1}{b^2+2b}+\sqrt{(1+a^2)(1+b^2)}\ge\frac{21}{4}$$

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$$\frac{1}{a^2+2a}+\frac{1}{b^2+2b}+\sqrt{(1+a^2)(1+b^2)}\overset{\text{Bergstrom}}{\ge}$$
$$\frac{4}{a^2+b^2+2(a+b)}+\sqrt{1+(a+b)^2-2ab+a^2b^2}\overset{a+b=ab}{=}$$
$$\frac{4}{a^2+b^2+2ab}+\sqrt{1+(a+b)^2-2(a+b)+(a+b)^2}=\frac{4}{x^2}+\sqrt{2x^2-2x+1}$$
$$\overset{?}{\ge}\frac{21}{4}\ (x=a+b)\Leftrightarrow\sqrt{2x^2-2x+1}\underbrace{\overset{?}{\ge}}_{(*)}\frac{21x^2-16}{4x^2}$$
$$\because a+b=ab\therefore 4(a+b)\le(a+b)^2\Rightarrow x\ge 4\rightarrow(1)$$
$$\therefore\frac{21x^2-16}{4x^2}=\frac{17x^2+4(x^2-16)}{4x^2}\ge\frac{17}{4}>0\therefore(*)\Leftrightarrow$$
$$(2x^2-2x+1)\overset{?}{\ge}\frac{(21x^2-16)^2}{16x^4}\Leftrightarrow 16x^4(2x^2-2x+1)\overset{?}{\ge}(21x^2-16)^2$$
$$\Leftrightarrow 32x^6-32x^5-425x^4+672x^2-256\overset{?}{\ge}0$$
$$\Leftrightarrow(x-4)\Big((x-4)(32x^4+224x^3+855x^2+3256x+13040)+52224\Big)\overset{?}{\ge}0$$
$$\rightarrow\text{true via }(1)\Rightarrow(*)\text{ is true}\therefore\frac{1}{a^2+2a}+\frac{1}{b^2+2b}+\sqrt{(1+a^2)(1+b^2)}\ge\frac{21}{4}$$
$$\forall\, a,b>0\,\big|\,a+b=ab,''=''\text{ iff } a=b=2\ (\text{QED})$$