ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=12$, then prove that :

$$\frac{1}{2a+3b+3c}+\frac{1}{2b+3c+3a}+\frac{1}{2c+3a+3b}\le 3$$

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$$\frac{1}{2a+3b+3c}+\frac{1}{2b+3c+3a}+\frac{1}{2c+3a+3b}=$$

$$\frac{1}{a+b+a+c+2(b+c)}+\frac{1}{b+c+b+a+2(c+a)}+\frac{1}{c+a+c+b+2(a+b)}$$

$$=\frac{1}{2x+y+z}+\frac{1}{2y+z+x}+\frac{1}{2z+x+y}\ (x=b+c, y=c+a, z=a+b)$$

$$=\frac{1}{x+y+z+x}+\frac{1}{y+z+x+y}+\frac{1}{z+x+y+z}=\frac{1}{Y+Z}+\frac{1}{Z+X}+\frac{1}{X+Y}\to(1)$$

$$(X=y+z, Y=z+x, Z=x+y)$$

Now, $X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0$

$\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y \Rightarrow X, Y, Z$ *form sides of a* triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding

$$2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\Rightarrow x=s-X, y=s-Y, z=s-Z$$

$$\therefore \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=12\Rightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=12$$

$$\Rightarrow\frac{1}{s-X}+\frac{1}{s-Y}+\frac{1}{s-Z}=12\Rightarrow\frac{\sum_{cyc}(s-Y)(s-Z)}{(s-X)(s-Y)(s-Z)}=12\Rightarrow\frac{4Rr+r^2}{r^2s}=12$$

$$\Rightarrow s=\frac{4R+r}{12r}\to(2)$$

$$\text{Now, via (1), }\frac{1}{2a+3b+3c}+\frac{1}{2b+3c+3a}+\frac{1}{2c+3a+3b}$$

$$=\frac{\sum_{cyc}(Z+X)(X+Y)}{(X+Y)(Y+Z)(Z+X)}=\frac{\sum_{cyc}X^2+3\sum_{cyc}XY}{2s(s^2+2Rr+r^2)}=\frac{\left(\sum_{cyc}X\right)^2+\sum_{cyc}XY}{2s(s^2+2Rr+r^2)}$$

$$=\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}\overset{\text{via (2)}}{=}\frac{5s^2+4Rr+r^2}{\left(\frac{4R+r}{6r}\right)(s^2+2Rr+r^2)}\overset{?}{\le}3$$

$$\Leftrightarrow(4R+r)(s^2+2Rr+r^2)-2r(5s^2+4Rr+r^2)\underset{(*)}{\overset{?}{\ge}}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}$$

$$(4R+r)(16Rr-5r^2+2Rr+r^2)-2r(5(4R^2+4Rr+3r^2)+4Rr+r^2)$$

$$=2r(16R^2-23Rr-18r^2)=2r(R-2r)(16R+9r)\overset{\text{Euler}}{\ge}0\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{1}{2a+3b+3c}+\frac{1}{2b+3c+3a}+\frac{1}{2c+3a+3b}\le 3$$

$$\forall\, a,b,c>0\ \Big|\ \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=12, \text{''}=\text{''}\ \text{iff } a=b=c=\frac{1}{8}\ \text{(QED)}$$