ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x(x+1) + y(y+1) + z(z+1) \le 18$, then prove that:

$$\frac{1}{x+y+1} + \frac{1}{y+z+1} + \frac{1}{z+x+1} \ge \frac{3}{5}$$

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$$\boxed{\text{Case 1}} \sum_{cyc} x^2 \ge 12 \text{ and then: } \frac{1}{x+y+1} + \frac{1}{y+z+1} + \frac{1}{z+x+1} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{9}{2\sum_{cyc} x + 3} \ge \frac{9}{36 - 2\sum_{cyc} x^2 + 3} \left(\because \sum_{cyc} x(x+1) \le 18 \Rightarrow \sum_{cyc} x \le 18 - \sum_{cyc} x^2 \right)$$

$$\overset{?}{\ge} \frac{3}{5} \Leftrightarrow \frac{3}{39 - 2\sum_{cyc} x^2} \overset{?}{\ge} \frac{1}{5} \Leftrightarrow 15 \overset{?}{\ge} 39 - 2\sum_{cyc} x^2$$

$$\left(\because -2\sum_{cyc} x^2 \ge 2\sum_{cyc} x - 36 \Rightarrow 39 - 2\sum_{cyc} x^2 \ge 2\sum_{cyc} x + 3 > 0 \right)$$

$$\Leftrightarrow \sum_{cyc} x^2 \overset{?}{\ge} 12 \rightarrow \text{true} \therefore \sum_{cyc} \frac{1}{x+y+1} \ge \frac{3}{5}$$

$$\boxed{\text{Case 2}} \sum_{cyc} x^2 \le 12 \text{ and then: } \frac{1}{x+y+1} + \frac{1}{y+z+1} + \frac{1}{z+x+1} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{9}{2\sum_{cyc} x + 3} \overset{\text{CBS}}{\ge} \frac{9}{2\sqrt{3\sum_{cyc} x^2} + 3} \overset{\sum_{cyc} x^2 \le 12}{\ge} \frac{9}{2\sqrt{3.12} + 3} = \frac{9}{15}$$

$$\therefore \sum_{cyc} \frac{1}{x+y+1} \ge \frac{3}{5} \text{ and so, combining both cases,}$$

$$\frac{1}{x+y+1} + \frac{1}{y+z+1} + \frac{1}{z+x+1} \ge \frac{3}{5}$$

$$\forall\, x, y, z > 0 \,\middle|\, x(x+1) + y(y+1) + z(z+1) \le 18 \text{ (QED)}$$