ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, ab + bc + ca = 3$ then:

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} \geq \frac{3}{4}$$

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$Lemma$:

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} \geq \frac{1}{1+xy} \ \forall\, x, y \geq 0$$

Proof:

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} - \frac{1}{1+xy} = \frac{xy(x-y)^2 + (xy-1)^2}{(1+x)^2(1+y)^2(1+xy)} \geq 0$$

Back to the problem:

$$LHS = \frac{1}{2}\sum\left[\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2}\right] \geq \frac{1}{2}\sum\frac{1}{1+ab} \overset{Bergstrom}{\geq} \frac{1}{2}\frac{9}{\sum ab + 3} = \frac{3}{4}$$