ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a^2 + \mathrm{b}^2 + \mathrm{c}^2 = 12$, then prove that :

$$\frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{\mathrm{b}^3+1}} + \frac{1}{\sqrt{\mathrm{c}^3+1}} \geq 1$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\sqrt{a^3+1}} \overset{?}{\geq} \frac{7-2a}{9} \Leftrightarrow \frac{1}{a^3+1} \overset{?}{\geq} \frac{(7-2a)^2}{81}$$

$$\left(\because a^2 < 12 \Rightarrow a < 2\sqrt{3} < \frac{7}{2} \Rightarrow 7 - 2a > 0\right) \Leftrightarrow (a^3+1)(7-2a)^2 - 81 \overset{?}{\leq} 0$$

$$\Leftrightarrow 4a^5 - 28a^4 + 49a^3 + 4a^2 - 28a - 32 \overset{?}{\leq} 0$$

$$\Leftrightarrow (a-2)^2(4a^3 - 12a^2 - 15a - 8) \overset{?}{\leq} 0 \Leftrightarrow 4a^3 - 12a^2 - 15a - 8 \overset{?}{<} 0$$

$$\Leftrightarrow (a-4)(2a+1)^2 - 4 \overset{?}{<} 0 \rightarrow \text{true} \because a < \frac{7}{2} < 4 \Rightarrow a - 4 < 0$$

$$\therefore \frac{1}{\sqrt{a^3+1}} \geq \frac{7-2a}{9} \text{ and analogs} \Rightarrow \frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{\mathrm{b}^3+1}} + \frac{1}{\sqrt{\mathrm{c}^3+1}} \geq \sum_{\text{cyc}} \frac{7-2a}{9}$$

$$= \frac{7}{3} - \frac{2}{9}\sum_{\text{cyc}} a \overset{\text{CBS}}{\geq} \frac{7}{3} - \frac{2}{9}.\sqrt{3\sum_{\text{cyc}} a^2} \overset{a^2+\mathrm{b}^2+\mathrm{c}^2=12}{=} \frac{7}{3} - \frac{2}{9}.6 = 1$$

$$\Rightarrow \frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{\mathrm{b}^3+1}} + \frac{1}{\sqrt{\mathrm{c}^3+1}} \geq 1$$

$\forall\, a, \mathrm{b}, \mathrm{c} > 0 \,\middle|\, a^2 + \mathrm{b}^2 + \mathrm{c}^2 = 12$, " = " iff $a = \mathrm{b} = \mathrm{c} = 2$ (QED)