ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca = 2abc$, then prove that :

$$\frac{a}{c(c+a)} + \frac{b}{a(a+b)} + \frac{c}{b(b+c)} \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{c(c+a)} + \frac{b}{a(a+b)} + \frac{c}{b(b+c)} = \frac{a+c-c}{c(c+a)} + \frac{b+a-a}{a(a+b)} + \frac{c+b-b}{b(b+c)} =$$

$$\frac{1}{c} + \frac{1}{a} + \frac{1}{b} - \frac{1}{c+a} - \frac{1}{a+b} - \frac{1}{b+c} = \frac{ab+bc+ca}{abc} - \sum_{cyc} \frac{1}{b+c} = 2 - \sum_{cyc} \frac{1}{b+c} \geq 1$$

$$\overset{ab+bc+ca=2abc}{\Leftrightarrow} \sum_{cyc} \frac{1}{b+c} \overset{(*)}{\leq} \frac{ab+bc+ca}{2abc}$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc} (s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (2) \therefore (*) \Leftrightarrow \sum_{cyc} \frac{1}{x} = \frac{s^2+4Rr+r^2}{4Rrs} \leq \frac{4Rr+r^2}{2r^2 s}$$

$$\Leftrightarrow 2R(4R+r) \geq s^2 + 4Rr + r^2 \Leftrightarrow s^2 \leq 8R^2 - 2Rr - r^2 \rightarrow \text{true}$$

$$\because s^2 \overset{\text{Gerretsen}}{\leq} 4R^2 + 4Rr + 3r^2 \overset{?}{\leq} 8R^2 - 2Rr - r^2 \Leftrightarrow 4R^2 - 6Rr - 4r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 2(R-2r)(2R+r) \overset{?}{\geq} 0 \rightarrow \text{true via Euler} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a}{c(c+a)} + \frac{b}{a(a+b)} + \frac{c}{b(b+c)} \geq 1$$

$$\forall a, b, c > 0 \,|\, ab + bc + ca = 2abc, '' ='' \text{ iff } a = b = c = \frac{3}{2} \text{ (QED)}$$