ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $ab+bc+ca+2abc=1$, then prove that :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-(a+b+c)\geq\frac{9}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$1=2abc+\sum_{cyc}ab\overset{A-G}{\geq}2abc+3.\sqrt[3]{a^2b^2c^2}\Rightarrow 2x^3+3x^2-1\leq 0$$

$$\left(x=\sqrt[3]{abc}\right)\Rightarrow(2x-1)(x+1)^2\leq 0\Rightarrow 2x\leq 1\Rightarrow\sqrt[3]{abc}\leq\frac{1}{2}\Rightarrow abc\leq\frac{1}{8}\to(1)$$

$$\text{Now, }\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-(a+b+c)-\frac{9}{2}=\frac{\sum_{cyc}ab-abc\sum_{cyc}a-\frac{9}{2}.abc}{abc}\geq$$

$$\frac{\sum_{cyc}ab-\frac{1}{3}\left(\sum_{cyc}ab\right)^2-\frac{9}{2}.abc}{abc}\overset{ab+bc+ca+2abc=1}{=}\frac{1-2t-\frac{(1-2t)^2}{3}-\frac{9t}{2}}{abc}\overset{?}{\geq}0\ (t=abc)$$

$$\Leftrightarrow 8t^2+31t-4\overset{?}{\leq}0\Leftrightarrow(8t-1)(t+4)\overset{?}{\leq}0\Leftrightarrow t=abc\overset{?}{\leq}\frac{1}{8}\to\text{true via }(1)$$

$$\therefore\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-(a+b+c)\geq\frac{9}{2}$$

$$\forall\,a,b,c>0\,\middle|\,ab+bc+ca+2abc=1,''='' \text{ iff } a=b=c=\frac{1}{2}\ \text{(QED)}$$