ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$ and $x + y \leq 2$, then prove that :

$$x^2(2-x) + y^2(2-y) + (x+y)\left(\frac{1}{xy} - xy\right) \geq 2$$

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$$x^2(2-x) + y^2(2-y) + (x+y)\left(\frac{1}{xy} - xy\right)$$

$$= 2(x^2+y^2) - (x^3+y^3) + \frac{x+y}{xy} - xy(x+y) \overset{2 \geq x+y}{\geq}$$

$$(x+y)((x+y)^2 - 2xy) - \left((x+y)^3 - 3xy(x+y)\right) + \frac{x+y}{xy} - xy(x+y)$$

$$= (x+y)^3 - 2xy(x+y) - (x+y)^3 + 2xy(x+y) + \frac{x+y}{xy} \geq \frac{4(x+y)}{(x+y)^2} \overset{x+y \leq 2}{\geq} \frac{4}{2}$$

$$\Rightarrow x^2(2-x) + y^2(2-y) + (x+y)\left(\frac{1}{xy} - xy\right) \geq 2$$

$$\forall\, x, y > 0 \,\middle|\, x + y \leq 2, '' ='' \text{ iff } x = y = 1 \text{ (QED)}$$