ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \in \mathbb{R}$. Prove that :

$$3^{|a|} + 3^{|b|} + 3^{|c|} \ge 3 + \sqrt{a^2 + b^2 + c^2}$$

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Let $f(x) = 3^x - x - 1 \ \forall \ x \ge 0$ and then : $f'(x) = (\ln 3).3^x - 1 \to (1)$

Now, $\because x \ge 0$ and $\ln 3 > 0, \therefore x.\ln 3 \ge \ln 1 \Rightarrow 3^x \ge 1$ and $\therefore$ via (1),

$f'(x) \ge \ln 3 - 1 > \ln e - 1 \Rightarrow f'(x) > 0 \therefore f(x)$ is $\uparrow$ on $[0, \infty) \Rightarrow f(x) \ge f(0) = 0$

$\forall \ x \in [0, \infty) \therefore 3^x - x - 1 \ \forall \ x \ge 0 \therefore 3^{|a|} \ge |a| + 1 \ \forall \ a \in \mathbb{R}$ and analogs

$$\therefore 3^{|a|} + 3^{|b|} + 3^{|c|} \ge 3 + \sum_{cyc} |a| \overset{?}{\ge} 3 + \sqrt{a^2 + b^2 + c^2} \Leftrightarrow \sum_{cyc} |a| \overset{?}{\ge} \sqrt{a^2 + b^2 + c^2}$$

$$\Leftrightarrow \sum_{cyc} a^2 + 2\sum_{cyc} |a||b| \overset{?}{\ge} \sum_{cyc} a^2 \Leftrightarrow \sum_{cyc} |a||b| \overset{?}{\ge} 0 \to \text{true}$$

$$\therefore 3^{|a|} + 3^{|b|} + 3^{|c|} \ge 3 + \sqrt{a^2 + b^2 + c^2} \ \forall \ a, b, c \in \mathbb{R}, '' ='' \text{ iff } a = b = c = 0 \ (\text{QED})$$