ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a+b+c=abc$, then prove that :

$$(a^2-1)(b^2-1)(c^2-1)\le\sqrt{(a^2+1)(b^2+1)(c^2+1)}$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\therefore abc=r^2s\to(2)\text{ and such substitutions}\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3),\ \sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via }(1)\text{ and }(3)}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4),$$

$$\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)\overset{\text{via }(1),(2)\text{ and }(3)}{=}(4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow\sum_{cyc}a^2b^2=r^2\left((4R+r)^2-2s^2\right)\to(5)$$

$$\text{Now, }(a^2+1)(b^2+1)(c^2+1)\overset{a+b+c=abc}{=}\left(a^2+\frac{abc}{\sum_{cyc}a}\right)\left(b^2+\frac{abc}{\sum_{cyc}a}\right)\left(c^2+\frac{abc}{\sum_{cyc}a}\right)$$

$$=\frac{abc}{\left(\sum_{cyc}a\right)^3}\left(a\sum_{cyc}a+bc\right)\left(b\sum_{cyc}a+ca\right)\left(c\sum_{cyc}a+ab\right)$$

$$\overset{a+b+c=abc}{=}\frac{abc}{a^3b^3c^3}\left(abc\left(\sum_{cyc}a\right)^3+a^2b^2c^2+abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)+\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}a^2b^2\right)\right)$$

$$\overset{\text{via }(1),(2),(4)\text{ and }(5)}{=}\frac{r^2s^4+r^4s^2+r^2s^2(s^2-8Rr-2r^2)+r^2s^2\left((4R+r)^2-2s^2\right)}{r^4s^2}$$

$$=\frac{16R^2}{r^2}\Rightarrow\sqrt{(a^2+1)(b^2+1)(c^2+1)}=\frac{4R}{r}\to(\mathrm{i})$$

ROMANIAN MATHEMATICAL MAGAZINE

$$\text{Again, } (a^2-1)(b^2-1)(c^2-1) = a^2b^2c^2 - 1 - \sum_{cyc} a^2b^2 + \sum_{cyc} a^2 \overset{a+b+c=abc}{=}$$

$$\left(\frac{\sum_{cyc} a}{abc}\right)^3 \left(a^2b^2c^2 - \left(\frac{abc}{\sum_{cyc} a}\right)^3 - \left(\sum_{cyc} a^2b^2\right)\left(\frac{abc}{\sum_{cyc} a}\right) + \left(\sum_{cyc} a^2\right)\left(\frac{abc}{\sum_{cyc} a}\right)^2\right)$$

$$\overset{\text{via } (1),(2),(4) \text{ and } (5)}{=} \left(\frac{s}{r^2s}\right)^3 \left(r^4s^2 - r^6 - r^4\left((4R+r)^2 - 2s^2\right) + r^4\left(s^2 - 8Rr - 2r^2\right)\right)$$

$$= \frac{4s^2 - 16R^2 - 16Rr - 4r^2}{r^2} \overset{\text{Gerretsen}}{\leq} \frac{4\left(4R^2 + 4Rr + 3r^2\right) - 16R^2 - 16Rr - 4r^2}{r^2}$$

$$= 8 \overset{\text{Euler}}{\leq} \frac{4R}{r} \overset{\text{via (i)}}{=} \sqrt{(a^2+1)(b^2+1)(c^2+1)}$$

$$\therefore (a^2-1)(b^2-1)(c^2-1) \leq \sqrt{(a^2+1)(b^2+1)(c^2+1)}$$

$$\forall\, a, b, c > 0 \,\middle|\, a + b + c = abc, '' ='' \text{ iff } a = b = c = \sqrt{3} \text{ (QED)}$$