ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $xyz = 1$, then prove that :

$$\left(\frac{x}{1+xy}\right)^2+\left(\frac{y}{1+yz}\right)^2+\left(\frac{z}{1+zx}\right)^2\geq\frac{3}{4}$$

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$$\left(\frac{x}{1+xy}\right)^2+\left(\frac{y}{1+yz}\right)^2+\left(\frac{z}{1+zx}\right)^2\overset{xyz=1}{=}$$

$$\left(\frac{x}{1+\frac{1}{z}}\right)^2+\left(\frac{y}{1+\frac{1}{x}}\right)^2+\left(\frac{z}{1+\frac{1}{y}}\right)^2=\left(\frac{xz}{z+1}\right)^2+\left(\frac{xy}{x+1}\right)^2+\left(\frac{yz}{y+1}\right)^2$$

$$\overset{xyz=1}{=}\left(\frac{1}{yz+y}\right)^2+\left(\frac{1}{zx+z}\right)^2+\left(\frac{1}{xy+x}\right)^2$$

$$\geq\left(\frac{1}{xy+x}\right)\left(\frac{1}{yz+y}\right)+\left(\frac{1}{yz+y}\right)\left(\frac{1}{zx+z}\right)+\left(\frac{1}{zx+z}\right)\left(\frac{1}{xy+x}\right)$$

$$=\frac{zx+z+xy+x+yz+y}{(xy+x)(yz+y)(zx+z)}\overset{xyz=1}{=}\frac{\sum_{cyc}xy+\sum_{cyc}x}{1+xyz+\sum_{cyc}xy+\sum_{cyc}x}\overset{?}{\geq}\frac{3}{4}$$

$$\Leftrightarrow\sum_{cyc}xy+\sum_{cyc}x\overset{?}{\geq}3+3xyz\overset{xyz=1}{=}6\rightarrow\text{true}\because$$

$$\sum_{cyc}xy+\sum_{cyc}x\overset{A-G}{\geq}3.\sqrt[3]{x^2y^2z^2}+3.\sqrt[3]{xyz}\overset{xyz=1}{=}6$$

$$\therefore\left(\frac{x}{1+xy}\right)^2+\left(\frac{y}{1+yz}\right)^2+\left(\frac{z}{1+zx}\right)^2\geq\frac{3}{4}$$

$$\forall\, x,y,z>0\,\big|\,xyz=1,''='' \text{ iff } x=y=z=1\ (\text{QED})$$