ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a^2 + b^2 + c^2 = \frac{3}{4}$, then prove that :

$$\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) \le -1$$

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Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \to (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s - x)(s - y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4)$$

Now, $\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) + 1 = \frac{(a-1)(b-1)(c-1) + abc}{abc}$

$$= \frac{\sum_{cyc} a - \sum_{cyc} ab + 2abc - 1}{abc} \le 0 \Leftrightarrow \sum_{cyc} ab + 1 \ge \sum_{cyc} a + 2abc$$

$$\overset{a^2+b^2+c^2=\frac{3}{4}}{\Leftrightarrow} \sum_{cyc} ab + \frac{4}{3}\sum_{cyc} a^2 \ge \left(\sum_{cyc} a\right) \cdot \sqrt{\frac{4}{3}\sum_{cyc} a^2} + \frac{2abc}{\sqrt{\frac{4}{3}\sum_{cyc} a^2}}$$

$$= \frac{\frac{4}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) + 2abc}{\sqrt{\frac{4}{3}\sum_{cyc} a^2}} \Leftrightarrow$$

$$\left(\frac{4}{3}\sum_{cyc} a^2\right)\left(\sum_{cyc} ab + \frac{4}{3}\sum_{cyc} a^2\right)^2 \ge \left(\frac{4}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) + 2abc\right)^2$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)\left(4\sum_{cyc} a^2 + 3\sum_{cyc} ab\right)^2 \ge 3\left(2\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) + 3abc\right)^2$$

$$\overset{\text{via (1),(2),(3) and (4)}}{\Leftrightarrow} (s^2 - 8Rr - 2r^2)\left(4(s^2 - 8Rr - 2r^2) + 3(4Rr + r^2)\right)^2 \ge 3\left(2s(s^2 - 8Rr - 2r^2) + 3r^2 s\right)^2$$

$$\Leftrightarrow 2s^6-(48Rr+30r^2)s^4+r^2(456R^2+372Rr+51r^2)s^2$$

$$-25r^3(4R+r)^3 \overset{(*)}{\geq} 0 \text{ and } \because 2(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to}$$

$$\text{prove } (*), \text{it suffices to prove : LHS of } (*) \geq 2(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (48R-60r)s^4-r(1080R^2-1332Rr+99r^2)s^2$$

$$+r^2(6592R^3-8880R^2r+2100Rr^2-275r^3) \overset{(**)}{\geq} 0$$

$$\text{and } \because (48R-60r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$

$$\text{to prove } (**), \text{it suffices to prove : LHS of } (**) \geq (48R-60r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (456R^2-1068Rr+501r^2)s^2 \overset{(***)}{\geq} r\begin{pmatrix}5696R^3-14160R^2r+8700Rr^2\\-1225r^3\end{pmatrix}$$

$$\text{Now}, 456R^2-1068Rr+501r^2=(R-2r)(456R-156r)+189r^2 \overset{\text{Euler}}{\geq} 189r^2$$

$$>0 \therefore \text{LHS of } (***) \overset{\text{Gerretsen}}{\geq} (456R^2-1068Rr+501r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(5696R^3-14160R^2r+8700Rr^2-1225r^3)$$

$$\Leftrightarrow 200t^3-651t^2+582t-160 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(74t^2+126t(t-2)+t+80) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)+1 \leq 0$$

$$\Rightarrow \left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right) \leq -1$$

$$\forall a,b,c>0 \,\middle|\, a^2+b^2+c^2=\frac{3}{4}, ''='' \text{ iff } a=b=c=\frac{1}{2} \text{ (QED)}$$