ROMANIAN MATHEMATICAL MAGAZINE

**If $a,b,c\geq 0$ and $a+b+c+abc=4$, then prove that :**
$$3(a^2+b^2+c^2)+13(ab+bc+ca)\geq 48$$

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$$4=\sum_{cyc}a+abc\overset{A-G}{\geq}3.\sqrt[3]{abc}+abc\Rightarrow t^3+3t-4\leq 0\ \left(t=\sqrt[3]{abc}\right)$$
$$\Rightarrow (t-1)\left(t^2+t+4\right)\leq 0\Rightarrow t=\sqrt[3]{abc}\leq 1\Rightarrow abc\leq 1\Rightarrow 4-abc\geq 3$$
$$\overset{a+b+c+abc=4}{\Leftrightarrow}\sum_{cyc}a\geq 3\rightarrow (1)$$
$$\text{Now, }3\left(a^2+b^2+c^2\right)+13(ab+bc+ca)=3\left(\sum_{cyc}a\right)^2+7\sum_{cyc}ab$$
$$\geq 3\left(\sum_{cyc}a\right)^2+7.\sqrt{3abc\sum_{cyc}a}\overset{a+b+c+abc=4}{=}$$
$$3\left(\sum_{cyc}a\right)^2+7.\sqrt{3\left(4-\sum_{cyc}a\right)\left(\sum_{cyc}a\right)}\overset{?}{\geq}48\ \left(4-\sum_{cyc}a=abc\geq 0\right)$$
$$\Leftrightarrow 7.\sqrt{3x(4-x)}\overset{?}{\geq}48-3x^2\ \left(x=\sum_{cyc}a\right)\Leftrightarrow 7.\sqrt{3x(4-x)}\overset{?}{\geq}3(4-x)(4+x)$$
$$\Leftrightarrow 7.\sqrt{x}\overset{?}{\geq}\sqrt{3(4-x)}(4+x)\Leftrightarrow 49x\overset{?}{\geq}3(4-x)(4+x)^2$$
$$\Leftrightarrow 3x^3+12x^2+x-192\overset{?}{\geq}0\Leftrightarrow (x-3)\left(3x^2+21x+64\right)\overset{?}{\geq}0\rightarrow \text{true}$$
$$\because x=\sum_{cyc}a\geq 3\ (\text{via }(1))\ \therefore 3\left(a^2+b^2+c^2\right)+13(ab+bc+ca)\geq 48$$
$$\forall\, a,b,c>0\,\middle|\, a+b+c+abc=4,''=''\ \text{ iff } a=b=c=1\ (\text{QED})$$