ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$, then:

$$\frac{a^3+b^3}{2}\le\left(\frac{a^2+b^2}{a+b}\right)^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Ravi Prakash-New Delhi-India

Put $a=r\cos\theta\,,b=r\sin\theta,r>0,0<\theta<\frac{\pi}{2}$

The given inequality can be written as

$$\frac{r^3}{2}(\sin^3\theta+\cos^3\theta)\le\frac{r^6(\cos^2\theta+\sin^2\theta)^3}{r^3(\cos\theta+\sin\theta)^3}$$

$$\Leftrightarrow(\sin\theta+\cos\theta)^4(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)\le 2$$

$$\Leftrightarrow(1+\sin 2\theta)^2\left(1-\frac{1}{2}\sin 2\theta\right)\le 2\Leftrightarrow(1+2\sin 2\theta+\sin^2 2\theta)(2-\sin 2\theta)\le 4$$

$$\Leftrightarrow 2+4\sin 2\theta+2\sin^2 2\theta-\sin 2\theta-2\sin^2 2\theta-\sin^3 2\theta\le 4$$

$$\Leftrightarrow\sin^3 2\theta-3\sin 2\theta+2\ge 0\Leftrightarrow(1-\sin 2\theta)^2(2+\sin 2\theta)\ge 0$$

which is true. Equality when $\theta=\frac{\pi}{4}$ or when $a=b$.

Solution 2 by Ravi Prakash-New Delhi-India

For $a,b>0$, consider

$$2(a^2+b^2)^3-(a+b)^3(a^3+b^3)$$

$$=2(a^6+3a^4b^2+3a^2b^4+b^6)-(a^3+3a^2b+3ab^2+b^3)(a^3+b^3)$$

$$=2a^6+6a^4b^2+6a^2b^4+2b^6-\left(a^6+3a^5b+3a^4b^2+2a^3b^3+3a^2b^4+3ab^5+b^6\right)$$

$$=a^6-3a^5b+3a^4b^2-2a^3b^3-3ab^5+3a^2b^4+b^6$$

$$=(a-b)^4(a^2+ab+b^2)\ge 0$$

Equality when $a=b$. Thus:

$$\frac{a^3+b^3}{2}\le\left(\frac{a^2+b^2}{a+b}\right)^3$$

Equality when $a=b$.