ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in \mathbb{R}$ and $a + b + c = 3$, then prove that :

$$\sqrt[3]{2a^2+6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6}\geq 6$$

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$\boxed{\text{Case 1}}$ Exactly two among a, b, c equal to zero : WLOG we may assume $b = c = 0$ $(a = 3)$ and then : $\text{LHS} = \sqrt[3]{24} + 2.\sqrt[3]{6} \approx 6.51874 > 6$

$\boxed{\text{Case 2}}$ Exactly one among a, b, c equals zero : WLOG we may assume $a = 0$ $(b, c > 0$ *with* $b + c = 3)$ and then : $\text{LHS} = \sqrt[3]{6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6} \overset{?}{>} 6$

$$\Leftrightarrow \left(\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6}\right)^3 \underbrace{\overset{?}{>}}_{(*)} \left(6-\sqrt[3]{6}\right)^3$$

Now, LHS of $(*) = 2b^2 + 6 + 2c^2 + 6$

$$+3.\sqrt[3]{(2b^2+6)(2c^2+6)}.\left(\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6}\right) \overset{\text{Jensen}}{\geq}$$

$$(b+c)^2+12+3.\sqrt[3]{4b^2c^2+6(b+c)^2+36}.2.\sqrt[3]{2\left(\frac{b+c}{2}\right)^2+6}$$

$$\left(\because f(x) = \sqrt[3]{2x^2+6}\ \forall\, x \in (0,3) \text{ is convex as } f''(x) = \frac{8(9-x^2)}{9(2x^2+6)^{\frac{5}{3}}} > 0\right)$$

$$\overset{b+c=3}{=} 21+6.\sqrt[3]{4b^2c^2+90}.\sqrt[3]{\frac{21}{2}} > 21+6.\sqrt[3]{45.21} \approx 79.87919$$

$$> \left(6-\sqrt[3]{6}\right)^3 (\approx 73.185667) \therefore \sqrt[3]{2a^2+6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6} > 6$$

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and then : $\sqrt[3]{2a^2+6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6} \overset{\text{Jensen}}{\geq}$

$$3.\sqrt[3]{2\left(\frac{a+b+c}{3}\right)^2+6} = 6 \therefore \text{combining all cases,}$$

$$\sqrt[3]{2a^2+6}+\sqrt[3]{2b^2+6}+\sqrt[3]{2c^2+6}\geq 6$$

$$\forall\, a, b, c \in \mathbb{R} \,\middle|\, a+b+c = 3, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$