ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c, d \in [1, 2]$, then prove that :

$$(a^2 + b^2 + c^2 + d^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}\right) \le 25$$

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Via Schweitzer (1914), for $0 < m < M$ and $x_i \in [m, M]$ $(i \in \overline{1, n})$,

$$\left(\frac{1}{n}\sum_{cyc} x_i\right)\left(\frac{1}{n}\sum_{cyc} \frac{1}{x_i}\right) \le \frac{(m + M)^2}{4mM} \to (1)$$

Choosing $n = 4, x_1 = a^2, x_2 = b^2, x_3 = c^2, x_4 = d^2$ and $\because a, b, c, d \in [1, 2]$

$$\therefore a^2, b^2, c^2, d^2 \in [1, 4] \Rightarrow m = 1, M = 4 \text{ and so, via } (1),$$

$$\left(\frac{a^2 + b^2 + c^2 + d^2}{4}\right)\left(\frac{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}}{4}\right) \le \frac{(1 + 4)^2}{4.1.4}$$

$$\Rightarrow (a^2 + b^2 + c^2 + d^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}\right) \le 25 \ \forall a, b, c, d \in [1, 2] \text{ (QED)}$$