ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c \in \mathbb{R}$ and $ab+bc+ca=3$, then prove that :

$$(2a^2+b^2+c^2)(2b^2+c^2+a^2)(2c^2+a^2+b^2) \geq 64$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly one among a,b,c equals zero : WLOG we may assume $a=0$ $(b,c>0$ *with* $bc=3)$ and then :

$$(2a^2+b^2+c^2)(2b^2+c^2+a^2)(2c^2+a^2+b^2)=(b^2+c^2)(2b^2+c^2)(2c^2+b^2)$$

$$=(b^2+c^2)\left(2(b^2+c^2)^2+b^2c^2\right)\overset{A-G}{\geq}2bc(9b^2c^2)=18.27>64$$

$$\Rightarrow(2a^2+b^2+c^2)(2b^2+c^2+a^2)(2c^2+a^2+b^2)>64$$

$\boxed{\text{Case 2}}$ $a,b,c>0$ and assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say); so $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\rightarrow(1)\Rightarrow a=s-x, b=s-y,$ $c=s-z \therefore abc=r^2s\rightarrow(2)$ and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow\sum_{cyc}ab=4Rr+r^2\rightarrow(3),\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (3)}}{=}$$

$$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\rightarrow(4),$$

$$\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)\overset{\text{via (1),(2) and (3)}}{=}(4Rr+r^2)^2-2r^2s.s$$

$$\Rightarrow\sum_{cyc}a^2b^2=r^2\left((4R+r)^2-2s^2\right)\rightarrow(5)$$

$$\text{Now, }(2a^2+b^2+c^2)(2b^2+c^2+a^2)(2c^2+a^2+b^2)\geq 64\overset{\because ab+bc+ca=3}{\Leftrightarrow}$$

$$\left(a^2+\sum_{cyc}a^2\right)\left(b^2+\sum_{cyc}a^2\right)\left(c^2+\sum_{cyc}a^2\right)\geq\frac{64}{27}\left(\sum_{cyc}ab\right)^3$$

$$\Leftrightarrow 27a^2b^2c^2+54\left(\sum_{cyc}a^2\right)^3+27\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)\geq 64\left(\sum_{cyc}ab\right)^3$$

$$\overset{\text{via (2),(3),(4) and (5)}}{\Leftrightarrow} 27r^4s^2+54(s^2-8Rr-2r^2)^3$$

$$+27r^2(s^2-8Rr-2r^2)\left((4R+r)^2-2s^2\right)-64(4Rr+r^2)^3\overset{(*)}{\boxed{\geq}}0 \text{ and}$$

$$\because 54(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\therefore\text{ in order to prove }(*)\text{, it suffices to prove :}$$

$$\text{LHS of }(*)\geq 54(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow(324R-297r)s^4-r(7668R^2-7938Rr+810r^2)s^2$$

$$+r^2\left(46496R^3-58440R^2r+14550Rr^2-1825r^3\right)\boxed{\overset{(**)}{\geq}}\,0$$

$and \because (324R-297r)\left(s^2-16Rr+5r^2\right)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$,

it suffices to prove : LHS of $(**) \geq (324R-297r)\left(s^2-16Rr+5r^2\right)^2 \Leftrightarrow$

$$\left(1350R^2-2403Rr+1080r^2\right)s^2\boxed{\overset{(***)}{\geq}}\,r\begin{pmatrix}18224R^3-34716R^2r+20535Rr^2\\-2800r^3\end{pmatrix}$$

Again, $\left(1350R^2-2403Rr+1080r^2\right)s^2 \overset{\text{Gerretsen}}{\geq}$

$$\left(1350R^2-2403Rr+1080r^2\right)\left(16Rr-5r^2\right)$$

$$\overset{?}{\geq} r\left(18224R^3-34716R^2r+20535Rr^2-2800r^3\right)$$

$$\Leftrightarrow 1688t^3-5241t^2+4380t-1300 \overset{?}{\geq} 0\ \left(t=\frac{R}{r}\right)$$

$\Leftrightarrow (t-2)\left((t-2)(1688t+1511)+3672\right) \overset{?}{\geq} 0 \rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**)$

$\Rightarrow (*)$ is true $\therefore \left(2a^2+b^2+c^2\right)\left(2b^2+c^2+a^2\right)\left(2c^2+a^2+b^2\right) \geq 64$ and so,

combining both cases, $\left(2a^2+b^2+c^2\right)\left(2b^2+c^2+a^2\right)\left(2c^2+a^2+b^2\right) \geq 64$

$\forall a,b,c \in \mathbb{R} \mid ab+bc+ca=3, ''='' $ iff $a=b=c=1$ (QED)