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If $a, \mathrm{b}, \mathrm{c} > 0$ and $a + \mathrm{b} + \mathrm{c} = 3$, then prove that :

$$\frac{1}{\sqrt{2a^2+1}}+\frac{1}{\sqrt{2\mathrm{b}^2+1}}+\frac{1}{\sqrt{2\mathrm{c}^2+1}}\ge\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\frac{1}{a}, \mathrm{y}=\frac{1}{\mathrm{b}}, \mathrm{z}=\frac{1}{\mathrm{c}} \text{ and then : } \sum_{\text{cyc}}\frac{1}{x} \overset{a+\mathrm{b}+\mathrm{c}=3}{=} 3 \to (1)$$

$$\left(\frac{1}{x}=a<3 \Rightarrow x>\frac{1}{3} \text{ and analogs}\right)$$

$$\text{We shall now prove that : } \frac{x^2}{\sqrt{x^2+2}} \ge \frac{5x-2}{3\sqrt{3}}\ \forall\, x\in\left(\frac{1}{3},\infty\right)$$

$$\text{If } \frac{1}{3}<x\le\frac{2}{5}, \text{ then : } \frac{5x-2}{3\sqrt{3}}\le 0<\frac{x^2}{\sqrt{x^2+2}} \text{ and so, we now focus on } x>\frac{2}{5}$$

$$\text{and then : } \frac{x^2}{\sqrt{x^2+2}}\ge\frac{5x-2}{3\sqrt{3}} \Leftrightarrow \frac{x^4}{x^2+2}\ge\frac{(5x-2)^2}{27} \Leftrightarrow 27x^4\ge\left(x^2+2\right)(5x-2)^2$$

$$\Leftrightarrow x^4+10x^3-27x^2+20x-4\ge 0 \Leftrightarrow (x-1)^2\left(x^2+4(3x-1)\right)\ge 0 \to \text{true}$$

$$\therefore \frac{x^2}{\sqrt{x^2+2}}\ge\frac{5x-2}{3\sqrt{3}}\ \forall\, x\in\left(\frac{1}{3},\infty\right) \text{ and analogs} \to (2)$$

$$\text{Now, } \frac{1}{\sqrt{2a^2+1}}+\frac{1}{\sqrt{2\mathrm{b}^2+1}}+\frac{1}{\sqrt{2\mathrm{c}^2+1}}=\sum_{\text{cyc}}\frac{1}{\sqrt{\frac{2}{x^2}+1}}=\sum_{\text{cyc}}\left(\frac{1}{x}\cdot\frac{x^2}{\sqrt{x^2+2}}\right)$$

$$\overset{\text{via }(2)}{\ge}\sum_{\text{cyc}}\left(\frac{1}{x}\cdot\frac{5x-2}{3\sqrt{3}}\right)=\frac{5}{\sqrt{3}}-\frac{2}{3\sqrt{3}}\cdot\sum_{\text{cyc}}\frac{1}{x}\overset{\text{via }(1)}{=}\frac{5}{\sqrt{3}}-\frac{6}{3\sqrt{3}}=\sqrt{3}$$

$$\therefore \frac{1}{\sqrt{2a^2+1}}+\frac{1}{\sqrt{2\mathrm{b}^2+1}}+\frac{1}{\sqrt{2\mathrm{c}^2+1}}\ge\sqrt{3}$$

$$\forall\, a,\mathrm{b},\mathrm{c}>0 \,\big|\, a+\mathrm{b}+\mathrm{c}=3, \text{"="} \text{ iff } a=\mathrm{b}=\mathrm{c}=1 \text{ (QED)}$$

Solution 2 by Tapas Das-India

Lemma:
If $x\in(0,3)$ then:

$$\frac{1}{\sqrt{2x^2+1}}\ge\frac{5-2x}{3\sqrt{3}}$$

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Proof:

$$\frac{1}{\sqrt{2x^2+1}} \ge \frac{5-2x}{3\sqrt{3}} \Leftrightarrow (5-2x)^2(2x^2+1) \le 27$$

$$\Leftrightarrow\ 8x^4-40x^3+54x^2-20x-2 \le 0 \Leftrightarrow (x-1)^2(4x^2-12x-1) \le 0$$

$$\Leftrightarrow (x-1)^2[(2x-3)^2-10] \le 0\ true.\ (as\ 2x-3<3)$$

$$\frac{1}{\sqrt{2a^2+1}}+\frac{1}{\sqrt{2b^2+1}}+\frac{1}{\sqrt{2c^2+1}} \overset{\text{Lemma}}{\ge} \sum_{\text{cyc}} \frac{5-2a}{3\sqrt{3}} = \frac{15-2\cdot 3}{3\sqrt{3}} = \sqrt{3}$$

Equality holds for $a=b=c=1$.