ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b} \in \mathbb{R}$ and $a^3 + \mathrm{b}^3 + a^2 + \mathrm{b}^2 = 4$, then prove that :

$$a^4 + \mathrm{b}^4 \geq 2$$

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$$a^3 + \mathrm{b}^3 + a^2 + \mathrm{b}^2 = 4 \Rightarrow (a+\mathrm{b})^3 - 3a\mathrm{b}(a+\mathrm{b}) + (a+\mathrm{b})^2 - 2a\mathrm{b} = 4$$

$$\Rightarrow x^3 - 3x\mathrm{y} + x^2 - 2\mathrm{y} = 4\ (x = a+\mathrm{b}, \mathrm{y} = a\mathrm{b}) \Rightarrow x^3 + x^2 - 4 = \mathrm{y}(3x+2)$$

$$\Rightarrow \frac{x^3 + x^2 - 4}{3x+2} \overset{(\bullet)}{=} \mathrm{y} \left(\text{if } 3x+2 = 0, \text{then} : x = -\frac{2}{3}, \text{but } x^3 + x^2 - 4 \neq 0 \text{ for } x = -\frac{2}{3} \Rightarrow 3x+2 \neq 0 \right)$$

$$\leq \frac{x^2}{4} \Rightarrow \frac{4(x^3 + x^2 - 4) - x^2(3x+2)}{4(3x+2)} \leq 0 \Rightarrow \frac{(3x+2)(x-2)\left((x+2)^2 + 4\right)}{(3x+2)^2} \leq 0$$

$$\Rightarrow \boxed{-\frac{2}{3} < x \leq 2} \to (1)$$

$$\text{Now}, a^4 + \mathrm{b}^4 = (a^2 + \mathrm{b}^2)^2 - 2a^2\mathrm{b}^2 = (x^2 - 2\mathrm{y})^2 - 2\mathrm{y}^2 = x^4 - 4x^2\mathrm{y} + 2\mathrm{y}^2$$

$$\overset{\text{via } (\bullet)}{=} x^4 - 4x^2 . \frac{x^3 + x^2 - 4}{3x+2} + 2\left(\frac{x^3 + x^2 - 4}{3x+2}\right)^2 \geq 2$$

$$\Leftrightarrow \frac{x^6 + 4x^5 + 2x^4 - 32x^3 + 2x^2 + 24x - 24}{(3x+2)^2} \leq 0$$

$$\Leftrightarrow x^6 + 4x^5 + 2x^4 - 32x^3 + 2x^2 + 24x - 24 \leq 0$$

$$\Leftrightarrow (x-2)\left(x^5 + 6x^4 + 14x^3 - 4x^2 - 6x + 12\right) \leq 0 \text{ and so, it suffices to prove :}$$

$$x^5 + 6x^4 + 14x^3 - 4x^2 - 6x + 12 \boxed{\overset{(*)}{>}} 0\ \left(\because x - 2 \leq 0 \text{ via } (1)\right) \text{ and}$$

$$\because x^4 - 4x^2 + 4 = (x^2 - 2)^2 \geq 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$x^5 + 5x^4 + 14x^3 - 6x + 8 > 0 \Leftrightarrow 81x^5 + 405x^4 + 1134x^3 - 486x + 648 > 0$$

$$\Leftrightarrow (3x-1)^2\left(9x^3 + 51x^2 + 159x + \frac{301}{3}\right) + \frac{1643}{3} - 43x \boxed{\overset{(**)}{>}} 0$$

$$\text{Now}, \frac{1643}{3} - 43x \overset{\text{via } (1)}{\geq} \frac{1643}{3} - 86 > 0 \Rightarrow \frac{1643}{3} - 43x > 0 \to (\text{i}) \text{ and also},$$

$$9x^3 + 6x^2 = 3x^2(3x+2) \geq 0 \to (\text{ii}) \left(\because 3x + 2 \overset{\text{via } (1)}{>} 0\right) \therefore (\text{i}), (\text{ii}) \Rightarrow$$

$$\text{in order to prove } (**), \text{it suffices to prove :}$$

$$45x^2 + 159x + \frac{301}{3} > 0 \Leftrightarrow 135x^2 + 477x + 301 > 0$$

$$\Leftrightarrow (3x+2)(15(3x+2) + 99) + 43 > 0 \to \text{true} \because 3x + 2 \overset{\text{via } (1)}{>} 0 \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true} \therefore a^4 + \mathrm{b}^4 \geq 2\ \forall\, a, \mathrm{b} \in \mathbb{R} \,\middle|\, a^3 + \mathrm{b}^3 + a^2 + \mathrm{b}^2 = 4, '' ='' \text{ iff } a = \mathrm{b} = 1\ (\text{QED})$$