ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$ then:

$$a^b b^a \le \left(\frac{a+b}{2}\right)^{a+b}$$

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$$a_i \ge 0 (i=1,2,\dots,n)$$

$$x_i > 0 (i=1,2,\dots,n), \sum_{i=1}^{n} x_i = 1$$

$$\sum_{i=1}^{n} x_i a_i \ge \prod_{i=1}^{n} a_i^{x_i} \quad (1)$$

Let: $a_1 = a, a_2 = b, x_1 = \frac{b}{a+b}, x_2 = \frac{a}{a+b}, x_1 + x_2 = 1$

$$\Rightarrow a^{\frac{b}{a+b}} b^{\frac{a}{a+b}} \le a\left(\frac{b}{a+b}\right) + b\left(\frac{a}{a+b}\right) = \frac{2ab}{a+b}$$

But: $\frac{2ab}{a+b} \le \frac{a+b}{2} \because \frac{2ab}{a+b} \le \frac{a+b}{2} \Leftrightarrow (a+b)^2 \ge 4ab \Leftrightarrow 4ab \Leftrightarrow (a-b)^2 \ge 0$

which is true for all $a,b>0$

$$\Rightarrow a^{\frac{b}{a+b}} b^{\frac{a}{a+b}} < \frac{a+b}{2} \Leftrightarrow \left(a^{\frac{b}{a+b}} b^{\frac{a}{a+b}}\right)^{a+b} \le \left(\frac{a+b}{2}\right)^{a+b} \Leftrightarrow a^b b^a \le \left(\frac{a+b}{2}\right)^{a+b}$$

Equality holds if and only if $a=b$.

* Prove (1)

$$\sum_{i=1}^{n} x_i a_i = a = a^{\sum_{i=1}^{n} x_i} = \prod_{i=1}^{n} a^{x_i} \Rightarrow (1) \Leftrightarrow \prod_{i=1}^{n} a^{x_i} \ge \prod_{i=1}^{n} a_i^{x_i} \Leftrightarrow \prod_{i=1}^{n} \left(\frac{a_i}{a}\right)^{x_i} \le 1$$

$$x \le e^{x-1} \forall x \Rightarrow \frac{a_i}{a} \le e^{\frac{a_i}{a}-1} \Rightarrow \left(\frac{a_i}{a}\right)^{x_i} \le e^{\frac{x_i a_i}{a} - x_i} \Rightarrow$$

$$\Rightarrow \prod_{i=1}^{n} \left(\frac{a_i}{a}\right)^{x_i} \le \prod_{i=1}^{n} e^{\frac{x_i a_i}{a} - x_i} \Leftrightarrow \prod_{i=1}^{n} \left(\frac{a_i}{a}\right)^{x_i} \le e^{\sum_{i=1}^{n}\left(\frac{x_i a_i}{a} - x_i\right)} = e^{1-1} = 1 \Rightarrow$$

$\Rightarrow$ (1) is true.